GRAPHICS

BY

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PREFACE.

These notes contain the substance of lectures on the subject of graphics which have been given to the students in Mechanical, Electrical and Chemical Engineering at the University of Pennsylvania for a number of years past.

They are intended to cover only fundamental principles, and those familiar with the subject will recognize that the methods of treatment used by many writers have been utilized in their preparation. Many of the short cuts in common use are not referred to in the text, as the time allotted to this work is limited, and while such short cuts are of special value in special work, they are readily grasped by one who has a good fundamental knowledge of the entire subject.

The treatment of trusses is short, but is believed to be full enough to enable beginners to grasp the general principles. The graphics of machines is limited to such examples as will most clearly set forth the effect of friction on the line of action of the forces to be determined. The latter portion of the book is devoted to forces not in the same plane, such as occur in many pieces of machinery.

It is intended that the book shall be used as a reference book, many numerical problems being worked out on the drawing board.

The writer is indebted to practically every author on the subject for methods or suggestions, and has simply made a continuous story of the subject matter.

University of Pennsylvania,
June 1, 1908.

GRAPHICS.

Equilibrium. A body is said to be in equilibrium when it is in the condition that is ordinarily understood by the expression "at rest," or when the body is moving with a uniform velocity.

A body cannot be in equilibrium under the action of a single force.

When two forces act on a body in equilibrium, the forces must be equal in amount, opposite in direction, and act in the same line. If the two forces are not equal in amount, they can be replaced by a single force, which will have the same effect on the body. If they are not opposite in direction, they will have a resultant single force. If they do not act in the same line, they can be replaced by a single force, or by a turning couple, or by both, and the body is not in equilibrium.

When a body is in equilibrium under the action of three forces, these forces must lie in the same plane, they must pass through the same point, and the force polygon must close.

Force Polygon. To draw a force polygon, suppose, in Fig. 1, a series of forces represented by **A, B, C** and **D** act at the point **O**, in the direction shown, the magnitude of the forces being as shown on the figure, 5, 4, 3 and 6 pounds. Begin at any point, a, and draw a line parallel to **OA** to the point b, making the distance ab equal to 5 pounds, to any scale, the line being drawn from a in the direction in which the force **AO** acts, as shown by the arrow head.

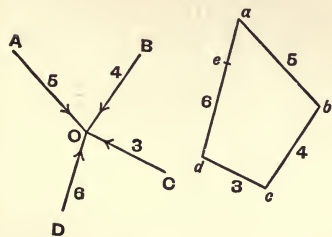Fig. 1.—Force Polygon.

From b, draw a line bc parallel to **BO**, and in length equal to 4 pounds to the same scale. Through c draw a line parallel to **CO**, and in length equal to 3 pounds to scale; and from d draw a line parallel to **DO**, and in length equal to 6 pounds to the same scale. This brings us again to the point a. This

diagram, *abcd*, is the force polygon, and, in this case, the force polygon closes.

Resultant Force. If, however, the force **D**, instead of reaching to the point *a*, extended only to the point *e*, the force polygon would not close, and the force *ea* would be required to close the polygon, or the force *ea* is the force necessary to hold the other forces in equilibrium, and the force *ae*, acting in the opposite direction to *ea*, is the resultant of the forces **A**, **B**, **C** and **D**.

Any Number of Forces. When any number of forces hold a body in equilibrium, the force polygon must close, and, in addition, the moment of all the forces about any point must be zero.

The statements already made about two and three forces are simply special cases of the above statements.

When any number of forces act on a body, the components of these forces acting along each of three lines at right angles to each other, such as the three converging edges of a box, must form closed force polygons if the body is in equilibrium, and, in addition, the moment of each set of components, taken about any point, must be zero.

Three Forces. Most of the problems in graphics relate to a condition of affairs in which three forces act at a point.

Referring to Fig. 2, if three forces, **AB**, **BC** and **CA**, act at the same point, one force only being known, and if these three
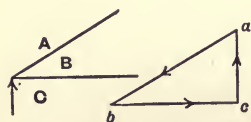

forces are in equilibrium, a force triangle can be drawn. First lay off *ca* from *c* upwards, proportional in length to the value of the force **CA**. Through *c* draw a line parallel to **CB**, and through *a* draw a line parallel to **AB**, and call the intersection of these two lines *b*.

Fig. 2.—Three Forces.

The force triangle for the forces acting at the point **ABC** is the triangle *abc*, and the amounts of the unknown forces are given by the sides of the triangle *bc* and *ab*.

Triangular Frame. Most of the structures treated of are built up of a series of triangles having forces acting at the vertices of each triangle. This form is used because external forces will not cause the structure to collapse and because the internal

force in each member can be accurately determined. It is assumed that the parts are held together by pins, and are not riveted together. Fig. 3 shows such a triangular frame. Assume that the external loads, acting on this frame, are as shown
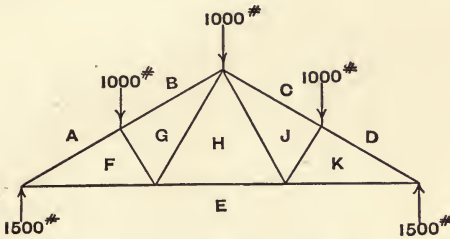

Fig. 3.—Triangular Frame.

at **AB, BC, CD, DE** and **EA.** As the forces acting at each vertex of this diagram are in equilibrium, it should be possible to draw a triangle of forces, or a closed polygon of forces at each point. In Fig. 4, the triangle *afe* determines the value of the forces acting at the left-hand corner, as the external forces are assumed to be known. At the point **ABGF**, the value of the forces **AB** and *fa* are known, **AB** being given, and the value of the force *fa* having been
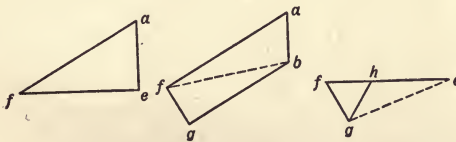

Fig. 4.—Separate Polygons.

determined from Fig. 4. Lay off the known forces in the direction in which they act, *faab*, and *fb* is the resultant of the known forces acting at this point. As these forces,

together with **BG, GF,** hold the point **ABGF** in equilibrium, draw through *b* a line parallel to **BG,** and through *f* a line parallel to **FG,** and call their intersection *g.* The force acting in **FG** is *fg* and the force acting in **BG** is *bg.*

Passing now to the vertex **GHEF,** the known forces are those acting in **EF** and in **FG.** Laying these forces down in the order in which they act, *eg* is the resultant of the known forces acting at this point. If through *e* a line is drawn parallel to **EH,** and through *g* a line drawn parallel to **GH,** and the intersection of these lines is called *h,* then *eh* is the force acting in **EH** and *gh* is the force acting in **GH.** Similar diagrams could be drawn for the vertexes, **BCJHG, CDKJ, JKEH,** and the right-hand corner.

It is not necessary to draw separate diagrams as shown in
Fig. 4, thereby drawing many of the lines twice, but one
diagram can be made to cover the entire truss.

Lettering Diagrams.   Before drawing a diagram of this
sort, the student should learn one consistent system of letter-
ing, the usefulness of which will appear later.   It will be
noticed in Figs. 3 and 4 that every external force and every
piece of the truss has been called by two letters, capital letters
being used on the truss itself, and small letters on the force
polygons.

In what is known as a right-hand system of lettering,
referring to Fig. 3, the external force on the left-hand corner is
called the force **EA**, and not the force **AE**.   The next force is
called the force **AB,** and not the force **BA.**   The force at the
right-hand end of the truss is called the force **DE** and not the
force **ED,** the letters being named clockwise, as though they were
the numbers on the face of a clock the center of which is in
the truss.

In drawing the force polygon, the arrows should be omitted,
and the forces should be laid off in such a way that the small
letters in the force polygon follow each other in the same order
that the large letters follow each other in the truss diagram,
that is, the force **AB** acts downward, and, in the force diagram,
ab acts downward.   The force *ea* in Fig. 4 acts from *e* to *a*, and
the letters follow each other in the order of **EA** in the truss
diagram.   As the external forces on the truss are in equilibrium,
first draw the force polygon for the external
forces, which must close.   Beginning with any
of the forces, as, for instance, with *bc* in Fig. 5,
mark the point *b*, then measure downwards,
because the force **BC** acts downwards, a dis-
tance *bc* equivalent to this force to scale.
From *c* mark downwards the force *cd;* from *d*
mark upwards the force *de;* from *e* mark
upwards the force *ea*, leaving *ab* downwards

Fig. 5.—Force
Diagram.

for the force **AB.**   Having now the value of the force *ea*,
through *e* draw a line parallel to **EF**, and through *a* draw a
line parallel to **AF,** and call the intersection *f*.   Through *f*,
draw a line parallel to **FG,** and through *b* a line parallel to **BG,**

and call the intersection of these two lines *g*. Through *g* draw a line parallel to **GH**, and through *e* a line parallel to **EH**, and call the intersection *h*. This Fig. 5 is then evidently Fig. 4 with the separate diagrams placed on each other, and the lengths of the lines *af*, *fg*, *gh*, *ef*, *eh*, and *gb* are the forces acting in the corresponding members of the truss. It is to be noted that one makes no distinction, in speaking, between *fg* and **FG**. The diagram having *fg* on it determines the amount and direction of the force, and the diagram having **FG** on it gives us the point of application of the force. The above operation, therefore, is simply the placing of one small letter on the force polygon after another in the order in which the large letters appear internally on the truss diagram.

Signs of the Forces. To determine the signs of the forces acting in the truss itself: When the external forces acting on a piece are such that the piece tends to elongate under the action of the forces, the piece is said to be "in tension," and the force is marked "plus." If the forces tend to shorten the piece, the piece is "in compression," and the force is marked "minus." It is evident that a force acting in a piece, such as **BG**, acts to the right at one end, and to the left at the other. In the same way, the piece can be named either the piece **BG** or the piece **GB**. By taking advantage of the fact that it can be named in either direction, the end of the piece under discussion can be definitely fixed. As the external forces have been named right-handed, the internal forces are named in the same way; that is, speaking about the vertex **ABGF,** the force in one piece is the force **BG,** and not the force **GB**. Speaking about the vertex **JHGBC**, this force in the same piece is called the force **GB,** reading in each case "clockwise" around the vertex.

To determine then, whether the piece **BG** is in tension or compression, first name it, and call it **BG**. As the letters **B** and **G** follow each other clockwise about the left end, this is the end under consideration, and the force **BG** tends to move the vertex **ABGF** in the direction from *b* to *g* in Fig. 5, or **BG** is in compression, and is minus.

If the piece had been called **GB**, the end under consideration would have been at the vertex **CJHGB**, and the force in **GB** would have tended to move that vertex in the direction from *g* to *b* in Fig. 5, or the piece **GB** is in compression, as before.

By a similar course of reasoning, **AF** is in compression; **GH** is in tension; **FG** is in compression; **EH** is in tension; **EF** is in tension.

Signs for Simple Loading. When a triangular truss is acted on by only three external loads, as shown in Fig. 6, the following course of reasoning will enable one to tell whether the members of the truss are in tension or in compression, wi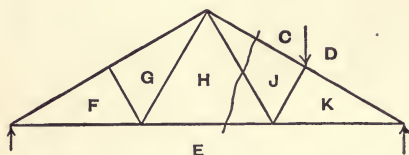thout drawing a diagram similar to Fig. 5. First, when three members of a truss act at one point and when two of these members are in the same line, and no external force acts at their junction, the third member of the truss going to the same vertex can have no stress in it. For instance, at **CGF**, **CG** and **FC** are in the same line. There is no external force at this vertex, and, therefore, there can be no stress in **GF**. If there is no stress in **GF**, there can be none in **GH**, because **EF** and **EH** are in one straight line. The active portion of the truss for this loading is then as shown by the heavy lines. The stress in **CF** must be minus, as it must have a vertical component downwards. The horizontal component of **CF** must be balanced by the horizontal force **FE**; therefore, **FE** must be in tension or plus.

Fig. 6.—Stress in any Member.

Similarly, **KD** must be in compression; **EK** in tension.

To determine whether **HJ** is in tension or compression, suppose the truss be cut off on the broken line through **C**, **J**, **H** and **E**. Then the forces acting are as shown in Fig. 7; the force **HE** acting in either one direction or the other, as shown, the forces **CJ** and **JH** acting in the same way. With these three forces, the portion cut off to the left is in equilibrium.

Fig. 7.—Stress in **HJ.**

The moment of all the forces acting on a body in equilibrium must be zero if taken about any point. The sign of the stress

in **HJ** is desired. If moments are taken about the point where **CJ** and **HE** intersect, the moments of these forces, **CJ** and **HE**, will be zero, and the moment of the force **EC** must balance the moment of the force **HJ**. The point **O** is the point about which these moments must be taken, and this is the right-hand end of the truss in this case. As the force **EC** tends to turn this portion of the truss around **O** clockwise, **HJ** must tend to turn it in the opposite direction, as it is in equilibrium. Therefore, the force acting in **HJ** must be in the direction of the arrow pointing downward, and the piece **HJ** must be in tension.

Similarly, to determine the sign of the stress acting in **CJ**, moments must be taken about the point **P**, as at that point the moments of **HJ** and **EH** are zero. The force **EC** tends to move the broken portion of the truss clockwise around **P**, and the force **CJ** must therefore tend to move it in the opposite direction, and the upper arrow represents the direction of the force in **CJ**, that is, **CJ** is in compression.

It has been shown that **HJ** is in tension. If this is the case, referring back to Fig. 6, then **HJ** has an upward vertical component at the lower end, which can only be counterbalanced by the force acting in **JK**, and **JK** must therefore have a component downwards at this lower end, and **JK** must be in compression.

The device of cutting the truss for the purpose of determining the sign of the force acting is a very simple one; and another example may tend to make the method clearer.

In Fig. 8, is shown a hipped truss, having a single force **CD** acting on it, in addition to the supports **DE** and **EC**. Gener-

Fig. 8.—Cutting a Truss.

ally under this character of loading each member of the upper chord **CF**, **CG**, **CJ**, and **DK** is in compression. The members of the lower chord **EF**, **EH**, and **EK** are in tension. To determine the sign of the stress acting in **FG**, cut the truss, as shown in the line I–I.
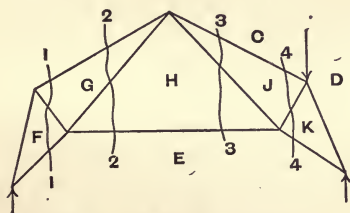
The portion of the truss to the left of the section being in equilibrium has the forces acting on it as shown in Fig. 9. As the sum of the moments of these forces is zero, and as the

stress is to be determined in **FG,** moments are taken about the point **Q,** where **EF** and **CG** intersect. The moment of **CG** and of **FE** is zero, and the moment of **EC** balances the moment of **FG. EC** tends to turn the portion of the truss clockwise around **Q. FG** must tend to turn it in the opposite direction, and, therefore, the force acting in **FG** must be downwards, and the piece is in tension.
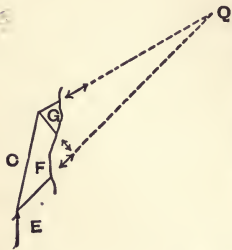

Fig. 9.—Stress in **GF.**

To determine the sign of the stress acting in **GH,** cut the truss in the line **2–2,** and in Fig. 10 this portion of the truss is reproduced with the forces acting on it. As the stress desired is that in **GH,** moments are taken about the point **R,** where the other unknown forces, **HE** and **CG** intersect. The force **EC** tends to turn the truss against the hands of the clock around the point **R;** therefore, the force **GH** must tend to turn it in the opposite direction, that is, the force in **GH** must be downwards, and **GH** must be in compression. To determine the sign of the stress acting in **HJ,** cut the truss on the line **3–3,** and again reproduce in Fig. 11 that portion of the truss to the left of the section. As the stress required is that in **JH,** moments are taken about the point where **CJ** and **HE** intersect at **S.**
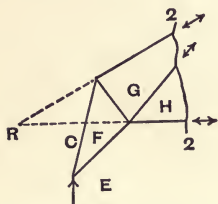
Fig. 10.—Stress in **GH.**
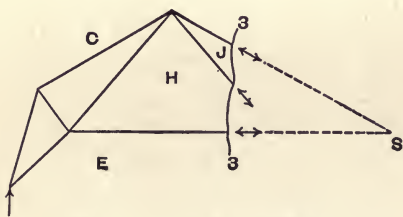
The force **EC** tends to turn the truss clockwise around **S;** therefore, **HJ** must be in tension.

To determine the stress in **JK,** cut the truss along the line **4–4.** Reproduce the portion of the truss to the left of the section, as shown in Fig. 12. As **KE** and **CJ** are the forces that are not required, moments are taken about the point where these lines intersect, at **T.** As the point **T** falls to the right of the

Fig. 11.—Stress in **HJ.**

line of action of the force **EC**, EC tends to turn the truss clock-
wise around the point **T.** The stress acting in **JK** must then
tend to turn it in the opposite
direction, and the force in **JK**
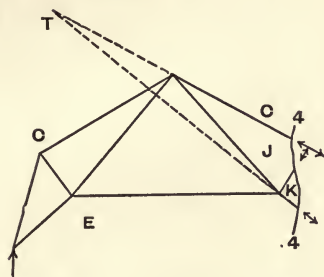must be upward, and **JK** must
be in tension.

Reactions. It has been taken
for granted that the forces acting
on the triangular frame are ver-
tical, and that the amount of
each of these forces is known.
Ordinarily these conditions are
not fulfilled. One or more forces

Fig. 12.—Stress in **JK**.

acting on the truss are given, and the direction of the forces
that hold them in equilibrium, and possibly the line of applica-
tion of the forces may be
known when the forces are
parallel. The simplest meth-
od of finding what are or-
dinarily called the reactions
is by calculation. Thus, in
a truss such as shown in
Fig. 13, having given the
forces as shown in that
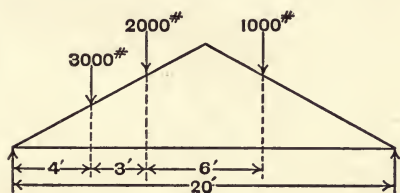
Fig. 13.—Reactions.

figure, the right reaction is

$$\frac{3000 \times 4 + 2000 \times 7 + 1000 \times 13}{20} = 1950 \text{ lbs.};$$

and the left reaction is the difference between the total load act-
ing downwards, 6000 lbs., and the right reaction, or 4050 lbs.

Equilibrium Polygons. When the loads acting on a piece are
not parallel, or when the line of application of one of the forces
is not known, the simplest method of getting the reaction or the
remaining force or forces is to draw what is known as an equili-
brium polygon. This is a ready device for reducing all the forces
acting on a piece to three, which must then pass through the
same point.

Thus in Fig. 14, suppose the forces **A, B, C** and **D** are given
as shown. To determine the point of application, direction and

amount of the resultant, first, draw the force polygon by laying off lines parallel and proportional to the forces, as shown at

Fig. 14.—Inclined Forces.

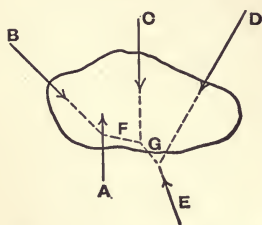$abcd$ in Fig. 15. The resultant force, or the equivalent of the forces shown in Fig. 14, takes the direction of the line e in Fig. 15 and is equal in amount to the length of e. To find its point of application, we can proceed as follows: Continue the force **A** and the force **B** until they intersect. The resultant of these two forces is f and acts at **F** in the direction shown. The resultant of **F** and **C** takes the direction g, and passes through the intersection of **F** and **C**. The four forces acting on the piece have been reduced to the force **D** and the force **G**, and the resultant of the four forces must then pass through the intersection of the forces **D** and **G**. As it has the direction of e, we have the line of action **E** and the amount and direction e of the resultant of **A**, **B**, **C** and **D**. The force that holds **A**, **B**, **C** and **D** in equilibrium acts in such a manner that the force polygon $abcde$ is closed. The lines f and g in this figure are auxiliary forces, put in for the purpose of conveniently reducing all the known forces to one.
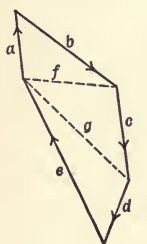
Fig. 15.—Resultant.

Equilibrium Polygon-Parallel Forces. If it had so happened that the forces **A**, **B**, **C** and **D** of Fig. 14 had been parallel, or nearly so, then another method of procedure would have been more convenient.
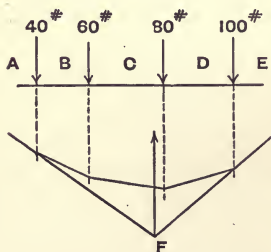
In Fig. 16 a series of parallel forces are shown, and it is required to determine, first, what single force will hold them in equilibrium. In Fig. 17, draw a line ab, equal to 40 lbs. bc, equal to 60 lbs., cd equal to 80 lbs., de equal to 100 lbs. Take any

Fig. 16.—Equilibrium Polygon.

point, o, and join it to each of the points, a, b, c, d and e. Now the force ab is equivalent to and can be replaced by bo and ao, and cd can be replaced by co and od, etc.

Going back to Fig. 16, through any point in the line of action of **AB**, draw one line parallel to *ao*, and another one parallel to *bo*. Where this latter line cuts the line of action of **BC**, draw a line parallel to *oc*. Where this line cuts the line of action of **CD** draw a line parallel to *od* across the space **D**. Where this line cuts the line of action of **DE**, draw a line parallel to *oe*. Thus, all the forces have been replaced by two, one parallel to *oa*, the other parallel to *oe;* and these two lines intersect at the point **F**. Therefore, the resultant of all these forces acts in a line through **F** and is parallel and equal to *ae*. The figure drawn below the loads in Fig. 16 is an equilibrium polygon.
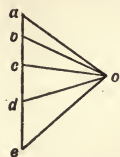
Fig. 17.
Force Diagram.

Three Parallel Forces. If a single force **AB**, Fig. 18, is given and it is known that the forces supporting **AB** are in the lines **BC** and **CA**, the amount of these forces can be determined as follows: Draw any triangle **I23** having its vertices on the line of action of the three forces. Laying off the amount of the force **AB** from *a* to *b* draw through *a* a line parallel to **I**, and through *b* a line parallel to **2**; where they intersect is *o*. Through *o*, draw a line *oc* parallel to **3**. Then *oc* will divide *ab* so that *bc* is the right reaction, and *ca* is the left reaction. **I23** is an equilibrium polygon. *o* is called the pole. The perpendicular distance *op* from *o* to *ab* is called the pole distance. What has been done in this figure is that **AB** has been divided into two forces, one *ob*, and the other *oa*. *ob* is **2**, and *oa* is **I**. The resultant of *ob* and *bc* is *oc* or **3**, that is, the resultant of **2** and the right reaction is **3** in the force polygon. The resultant of *oa* and *ac* is **3**, or the resultant of **I** and the left reaction is **3**.
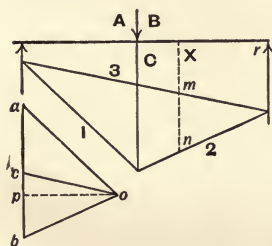
Fig. 18.—Parallel Forces.

Ordinarily, when only three forces are in question, it is not necessary to draw the force polygon separated from the truss polygon. Thus, in Fig. 19, the same figure has been drawn, putting the force polygon at each of the three places that it might have been drawn on the figure, on the right and left and

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near the center of the figure.   The right reaction is from b to c, and the left reaction is from c to a.   In laying out these figures, the order in which the letters are read does not tell in which direction the force acts.
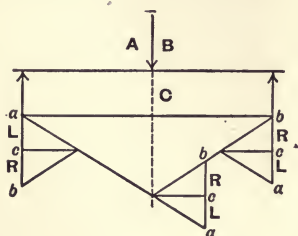

Fig. 19.—Location of Force Polygon.

Equilibrium Polygon a Bending Moment Diagram.   The equilibrium polygon is also a bending moment diagram.   At any point **X**, Fig. 18, the moment of the right reaction is $bc \times r$**X**, and this is the bending moment at **X**.

The two triangles obc and $mn23$ have the sides parallel and are similar.

We can write as the bases and altitude are proportional

$$\frac{mn}{bc} = \frac{\mathbf{X}r}{op} \quad \text{or}$$

$$bc \times \mathbf{X}r = op \times mn$$

The left side of this equation is the bending moment at **X** and therefore the bending moment is the pole distance (op) times the height (mn) of the equilibrium polygon.

Oblique Forces.   Referring again to Figs. 14 and 15 which are partly reproduced in Fig. 20, the forces **A**, **B**, **C** and **D** are drawn on the body, and also on the force polygon.   Taking any point o, either inside or outside of the force polygon, as a pole, draw the rays from o to the corners of the force polygon.   Draw a line from any point in **A** parallel to **I**, and from the same point a line parallel to **2**.   Where this latter line cuts **B**, draw a line parallel to **3**.   Where this line cuts **C**, draw a line parallel to **4**.   Where this line cuts **D**, draw a line parallel to **5**.   Then, the four forces, **A**, **B**, **C** and **D** are the equivalent of the forces **I** and **5**.

Fig. 20.—Equilibrium Polygon for Inclined Forces.

The force required to hold these forces in equilibrium passes through the intersection of **I** and **5**.   The amount of this force is *e*, and the direction in which the force acts is shown by the arrow and is parallel to *e*.   The forces **A, B, C, D** and **E** are in equilibrium; and the polygon that we have drawn, **12345,** is called an equilibrium polygon.   When the forces are in the same plane, and the equilibrium polygon closes, the resultant moment of the forces is zero, because the forces can be reduced to three, passing through one point.

When a number of forces act on a body in the same plane, if the the body is in equilibrium, first, the force polygon closes, second, the equilibrium polygon closes.

Referring to Fig. 16, if instead of the forces shown being balanced by a single force **F**, they are to be supported by two forces, the amount of these forces can be determined as follows: Fig. 21 is a reproduction of Fig. 16 and part of Fig. 17.

Draw the lines of action, **I,** **I** of the two supporting forces. The sides of the equilibrium polygon terminating in **F** stop at the points where they cross the lines of action of the two desired forces, **I, I.** Joining these points by a straight line **F'**, gives us an equilibrium polygon with an additional side. The line drawn through *o* parallel to this side and ending in *f'* will give the two reactions, *ef'*

Fig. 21.—Changing Supports.

and *f'a*, which support the four given loads, and *ef'* is the right reaction, and *f'a* is the left reaction.   If the supports are to be at **2–2**, the closing line of the equilibrium polygon is the line **F''**, and the two reactions are found by drawing *of''* parallel to **F''**, giving *ef''* and *f''a* for the right and left reactions.

Checking Graphical Work. An independent method of checking graphical work is always desirable.   The simplest method of checking work such as is shown in Figs. 3 and 5 is to continue the work until all the letters are on the board, and when this is the case, there are three separate lines which will

determine $k;$ a line through d parallel to **DK**, a line through e parallel to **EK**, and a line through j parallel to **JK**. If these lines all pass through the same point, then the graphical work is well checked.

By Calculation. With a symmetrical truss, it is not necessary to carry the work this far, and the simplest method of checking the work is to determine the stress in **HE** by calculation, after having determined it graphically, cutting the truss as shown in Fig. 7.

To get the stress in **HE**, take moments about the top vertex. The moments of all the loads to the left of the vertex, divided by the height of the truss, should give the stress in **HE**. A similar method may, if desired, be used for determining the stress in any single member of the truss, that is, cut the truss through three members, one of which is the desired one. Take moments about the point at which the other two intersect, and the moment of the desired force about this point is the moment of all the other external forces on one side of the break in the truss.

Maximum and Minimum.   When more than one set of forces act on a truss either at the same time, or in combinations, it is desirable to know the maximum stress which may come in a member, and the minimum stress.

Suppose a truss of the type shown in Figs. 3 or 8 is to be used for a roof. The weight of the truss and its covering is a permanent load which must be carried by the truss, and is called the "dead" load. There may be, at times, a snow load on the roof, or a wind load acting on one side or the other. Each of these loads must be treated independently and the stress brought in each member by the dead load, snow load, wind on the right and wind on the left determined independently and tabulated. The maximum stress is the maximum combination, numerically, which can occur, and which may be made up of the stresses due to the dead load, snow load, and the wind on one side. The minimum stress in a particular member always contains the dead load stress, and, in addition to this, may contain all the stresses of the opposite sign which might occur simultaneously. Thus, if it were possible in a series of members to have the stresses of the amount and signs as shown in the table, the maximum, in

each case, is indicated at the bottom, and the minimum directly below it.

Member.....	AB	BC	CD	DE
Dead...........	+3000	−3800	−3800	+ 800
Snow..........	−1600	−1300	−3000	+ 400
Wind on r......	+ 800	+2000	+4500	+2000
Wind on l.....	−1200	− 300	−1600	−1000
Maximum.....	+3800	−5400	−8400	+3200
Minimum.....	+ 200	−1800	+ 700	− 200

It will be noticed that the maximum value is the maximum possible combination numerically, whatever may be its sign, but always including the dead load; and the minimum is the combination which is algebraically farthest from the maximum, whether of the same sign or opposite sign, but always containing the dead load.

Approximate Loads. To determine the probable loads coming on a frame of this sort, a formula is generally given, showing the relation between the span and the distance the trusses are apart, and giving the total weight of the truss. The approximate weight of the truss, whether wood, steel, or a combination, may be taken from the formula $W = \dfrac{S}{25} + \dfrac{S^2}{6000}$, in which S is the span in feet and W is the weight of the truss in

Fig. 22.—Distribution of Loads.

pounds per square foot of horizontal projection of the roof.* The load is then supposed to be carried at the apexes on the top chord, and divided in proportion to the span. Thus, in Fig. 22, following the formula given, the total weight of the truss is found by multiplying th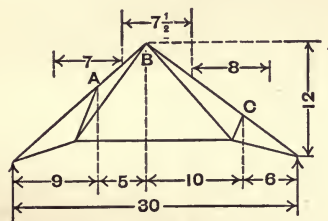e quantity found, from the above formula, by the span and by the distance between the trusses, both in feet. As the total span is thirty feet, the part of the weight coming on

* Ricker, Bulletin No. 16, University of Illinois Engineering Experiment Station.

the apex **A** is $\dfrac{7}{30}$ of the weight, on the apex **B** is $\dfrac{7.5}{30}$, and on the

apex **C** is $\dfrac{8}{30}$ of the weight.   The portions of the weight acting

over the reactions are $\dfrac{4.5}{30}$ and $\dfrac{3}{30}$, and these are not included in

the reactions in determining the stress in the various members
of the truss.

The roof covering is ordinarily taken at so many pounds
per running foot on the top chord, varying with the type of cover-
ing and sheathing used.   Thus, in Fig. 22, the total amount of
sheathing is that covering a surface represented by the length
of the rafters in one direction, and the distance between the
trusses in the other.   In this particular case, the left rafter has
a length equal to the $\sqrt{340}$; and the right a length equal to the

$\sqrt{400}$.   The load coming on the apex **A** is $\dfrac{7}{14}$ of the load on

the left rafter; on the apex **B** it is $\dfrac{2.5}{14}$ of the load on the left

rafter plus $\dfrac{5}{16}$ the load on the right rafter.   On the apex **C** the

load is $\dfrac{8}{16}$ the load on the right rafter.   An approximate value of

the weight of covering per square foot may be taken at 12 lbs. per
square foot for the purpose of this work, but for any actual truss
should be calculated from the weight of the actual materials to
be used.

Snow.   The snow load is usually taken at so many pounds
per square foot of the horizontal projection of the roof, and,
depending on the location of the structure, should have a load
per square foot equivalent to the probable maximum snowfall
in the locality.   For this work it may be taken as 30 lbs. per
square foot.   This load is supposed to be uniformly distributed
over the horizontal projection of the rafter. . Thus the total
snow load on this roof is 30 times the distance between the rafters

times the load per square foot, and of this load $\dfrac{7}{30}$ is carried at

A, $\dfrac{7.5}{30}$ at **B**, and $\dfrac{8}{30}$ at **C.**

When the slant of the roof is such that the snow will not lie, that portion of the roof is supposed to have no snow load. The angle at which the snow will not lie on the roof is assumed to be about 60 degrees.

Referring back to Fig. 8, if the angle of **DK** or **FC** is 60 degrees or more, the snow will lie only on the rafters **CG** and **CJ**. One-half the entire snow load will come at the center apex. One-half the load on **CG** will come at the left apex, and one-half the load on **CJ** will come at the right apex.

Wind. The wind is supposed to act at right angles to the roof, and the amount varies with the inclination of the roof. One-half of the load on each rafter is taken to the adjacent apex, so that in Fig. 8 again, at the apex **CDKJ**, one-half the wind load on **JC** acting perpendicular to **JC**, is supposed to be concentrated, together with one-half the wind load on **DK**, acting at right angles to **DK**. The normal wind load in pounds per square foot may be taken as $\frac{2}{3}$ the inclination of the rafter in degrees up to 45° and, above that inclination, may be assumed at 30 lbs. If the horizontal wind pressure exceeds thirty pounds, increase the normal pressure proportionately.

One End Free. When the length of the truss is such that the expansion and contraction of the truss, due to changes in temperature, would bring horizontal forces between the walls supporting the roof, one end of the truss is carried on rollers in such a way that the reaction at that end is vertical. The reaction at the other end is then at such an angle that it passes through the point in which the resultant of the wind forces intersects the vertical reaction at the free end.

To determine the line of action of the reaction at the fixed end, one can draw an equilibrium polygon between the known forces, and thereby determine an additional point in the line of action of the reaction at the fixed end. In doing this, the fixed end of the truss should be the starting point of the equilibrium polygon. It is probably quite as easy to determine the resultant of the wind forces that are known, and find where the line of action of this resultant cuts the reaction at the free end, and this is the point through which the reaction at the fixed end must pass.

Wind on Fixed End.   Thus, in Fig. 23, if the left end is free, the reaction at this end then will be vertical.   The forces acting on the roof when the wind is on the right are **A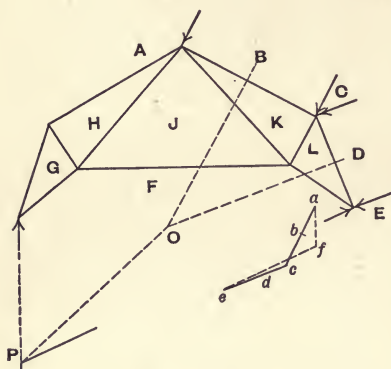B**, **BC**, **CD**, **DE**, **EF**, and **FA**, the latter two being unknown.   Through the center of the member **BK** draw a line at right angles to **BK** and through the center of the member **DL**, draw a line at right angles to **DL**, intersecting the line just drawn at **O**.   Draw the force polygon *abcde*, and through **O** draw a line **OP** parallel to *ea*.   The point **P** is the point through which the right reaction must pass.   The direction of the force **EF** is then known, and the force in each member of the frame may be determined.

Fig. 23.—Wind on Fixed End.

Wind on Free End.   When the wind is on the other side of the truss, the forces are as shown in Fig. 24.   Through the middle of **BG** and **DH** draw lines at right angles to **BG** and **DH** intersecting at **O**.   Draw the force polygon *abcde*, and through **O** draw a line parallel to *ea*.   This, then, is the line of action of the resultant wind forces, and intersects the force **FA** in the point **P**.   Drawing from **P** to the right support gives **EF** for the direction of the right reaction, and data enough have been obtained to determine the stress in each member of the truss.   It will be seen that the stresses brought in the members of the truss of this type differ with the wind on the different sides, while with a symmetrical truss, and both ends fixed, a single diagram will determine the stress in all members, with the wind on either side.
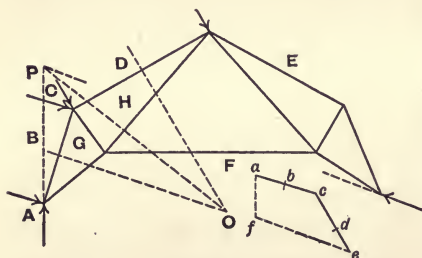
Fig. 24.—Wind on Free End.

Equal Triangles. When a frame is made up of equal triangles, it is not necessary to draw diagrams to scale to determine the stresses that exist in each member, as simple calculations will ordinarily determine the amount, and inspection will give the sign of the stress. For instance in Fig. 25, suppose the truss to be of the shape shown (a Warren truss), made up of equal triangles. It is convenient in a diagram of this sort to use letters, instead of actual loads, and to put in the actual value of the load later. In the figure, the triangles are supposed to be exactly alike. Suppose the distance marked **B** to be $\dfrac{1}{12}$ the length of the truss; the height of the truss to be **A**; and the inclined side of each triangle to be **C**. If the force acting vertically at the left reaction is called a, and the horizontal and inclined components are called b and c; then the proportions of a, b, and c are the same as those of **A**, **B**, and **C**.

Fig. 25.—Truss of Equal Triangles.

For convenience in using, so that fractional portions of the load do not occur, the apex load is called such a number that the reactions in each case will be whole numbers; that is, if the load is put on the top chord at **I**, and the load is called $6a$, the reactions are $5a$ and a. If the load is put on the point **2**, the reactions are $4a$ and $2a$. If put on the point **3**, the reactions are $3a$ and $3a$, etc. If, however, the loads are put on the lower chord, as at **I′**, the load is called $12a$, and the left reaction then will be $11a$ and the right reaction a. If the load $12a$ is put on **3′** the left reaction will be $9a$, and the right reaction $3a$. If the load is put on **5′**, the left reaction will be $7a$, and the right reaction $5a$.
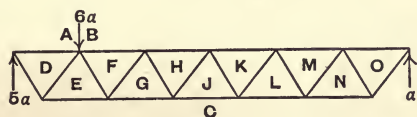

Fig. 26.—Load at One Point.

Inclined Members. It is required to determine the stress in the inclined members of this truss, with a load of $6a$ at the point **I**. In Fig. 26, each member of the truss has been lettered,

and, beginning at the left-hand end, the stress in **CD** will be + 5*c*. The stress in **DE**, which must carry the vertical component of the stress in **DC**, will be −5*c*. As the vertical component of 5*c* is 5*a*, and as **DE** is in compression, evidently **DE** supports 5 of the 6*a* of the load **AB**, and **EF** must then be in compression to the amount of −1*c*. **FG** must be in tension to the same amount, and the pieces alternate in tension and compression going to the right, and the final member **OC** must be in tension to the amount of *c*, as it must support the reaction *a* at the right end. It is evident, then, that starting from either end, the inclined members of this truss are alternately in tension and compression, until they reach the members supporting the load. Here both are in compression.

If, in addition to the load at **AB**, an additional load is placed on the next panel point, as **BP**, Fig. 27, the left reaction will be 5*a* + 4*a*, and the right reaction will be *a* + 2*a*. The stress in any member, as, for instance, **EF**, will be + 3*c*. The stress in any member as **JK**, will be −3*c*.
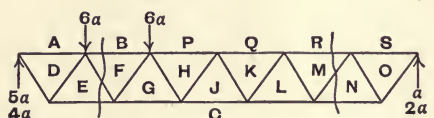
Fig. 27.—Load at Two Points.

Evidently, the member **EF** must support the vertical component of all the loads to the left of **EF**, so that if the truss is cut as shown by the broken line, the figure will be as shown in Fig. 28, to the left of the section, and this piece is in equilibrium. **BF** and **EC** have no vertical component; therefore, the vertical component of **EF** must hold in equilibrium the vertical component of **CA** and **AB**. **CA** is 9*a* acting upwards; **AB** is 6*a* acting downwards, and the resultant of these is 3*a* acting upwards; therefore, the force in **EF** must have a vertical component of 3*a* acting downwards, and **EF** must be in tension, and have in it the force of + 3*c*, which is the inclined component of 3*a*.

Fig. 28.—Stress in Inclined Member.

Similarly, if the truss is cut in the line **MN**, the portion of the

truss to the right of this section, shown in Fig. 28, is in equilibrium and the vertical component of **MN** must hold in equilibrium the vertical component of **CP.** **CP** is $3a$ acting upwards; therefore, **MN** must have tension in it of $+3c$, as its vertical component must be $3a$ acting downwards.

Horizontal Members. To determine the stress in the horizontal members, take moments of the forces to the left of the section, after cutting it, as shown in Fig. 27 and Fig. 28. Thus to determine the stress in **EC,** take moments about the point where **EF** and **FB** intersect. The moment of the force $9a$ acting upwards must be equal to the moment of the force **EC.** As the force **CA** tends to turn the portion of the truss clockwise around the point **ABFED,** and as the stress in **EC** must tend to turn it in the opposite direction, **EC** must be in tension.

To determine the amount of the stress in **EC** :—

$$9a \times 2\mathbf{B} = \mathbf{CE} \times \mathbf{A}$$

or,
$$\mathbf{CE} = 2 \times 9a \times \frac{\mathbf{B}}{\mathbf{A}}.$$

But $a \times \dfrac{\mathbf{B}}{\mathbf{A}} = b;$ therefore, the stress in $\mathbf{CE} = 18b$.

To determine the stress in **BF,** take moments about the point in which **EF** and **EC** intersect, and the stress in $\mathbf{BF} = (3 \times 9 - 6)\, b = -21b$.

Similarly, referring to Fig. 27, the stress in **QK** is found by taking moments of the forces on the right about the point **JKLC,** and is equal to $(3 \times 5)b = -15b$, or, taking moments from the left-hand end :—

$$(9 \times 7 - 6 \times 5 - 6 \times 3)\, b = -15b.$$

The sign in each case is written before the final answer, knowing from inspection whether the piece is in compression or in tension.

A Uniform Moving Load. If a force of $6a$ per panel point, Fig. 27, starts at the left-hand side of the truss, it is evident that in any inclined member, as **JK,** the stress increases until all the panel points to the left of **JK** are covered, and, in this particular case, is negative. If, now, an additional panel point is

covered, the stress in **JK** becomes less, because the right reaction has increased by a portion of the panel point load only, and to determine the stress in **JK**, the entire panel point load that has passed **JK** must be taken off; therefore, the maximum live load stress comes in **JK** when part of the truss only is loaded with a uniform panel point load. *As the maxium stress in* **JK** *depends only on the number of panel points that are loaded, the maximum stress in* **JK** *will occur when the long part of the span is loaded, and the minimum will occur when the short part of the span is loaded.*

In the particular case under consideration, if the panel points **AB**, **BP** and **PQ** are loaded, the stress in **JK** is minus, whereas, if **QR** and **RS** only are loaded, the stress in **JK** will be positive.

With the long part of the span loaded, the right reaction is $1 + 2 + 3 = 6a$, and the maximum stress in **JK** is $- 6c$. With the short end loaded, the left reaction is $a + 2a$ and the stress in **JK** is $+ 3c$. Therefore, as the maximum stress in **JK** is $-6c$,

Fig. 29.—Maximum Chord Stresses.

and the minimum is $+ 3c$, had the whole truss been loaded, the stress in **JK** would be $-6 + 3 = -3c$. The weight of a truss of this sort can therefore be treated as though it were a uniform live panel point load, covering all the points of the truss.

To determine the maximum stress in the chord members, take, for instance, the piece **JC**, Fig. 29. With the load at the first panel point only, **AB**, the stress in **JC** is

$$(5 \times 6 - 6 \times 4)\, b = + 6b, \text{ or } (1 \times 6)\, b = + 6b.$$

With panel points **I** and **2** loaded, the stress is
$$[(5 + 4) \times 6 - 6 \times 4 - 6 \times 2]b = + 18b, \text{ or } [(1 + 2) \times 6]b = + 18b.$$

With panel points **AB**, **BP** and **PQ** loaded, the stress in **JC** is
$$[(5 + 4 + 3) \times 6 - 6 \times 4 - 6 \times 2]b = + 36b, \text{ or } (1 + 2 + 3)6b = + 36b.$$

An additional load placed on **QR** increases the left reaction, and therefore the stress in **JC**, and the maximum stress in **JC** will then come when the entire structure is loaded, that is, when there is a load at each panel point.

When this is the case, the diagram, Fig. 29, represents the loading, and the maximum stress in **PH**, for instance, is $(15 \times 5 - 6 \times 3 - 6 \times 1)b = -51b$. The maximum stress in **LC,** for instance, is $(15 \times 4 - 6 \times 2)b = +48b$, the sign in each case being placed before the final answer, as the top chord is in compression, and the bottom in tension.

Pratt Truss. Suppose the truss to be made as shown in Fig. 30, instead of as in the preceding figures, and that the load is on the bottom chord. The truss shown has 8 panels, therefore call
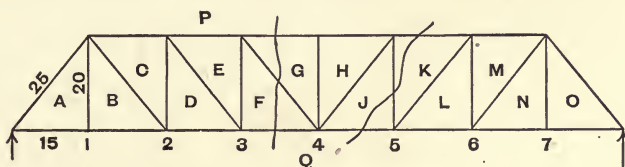
Fig. 30.—Pratt Truss.

the panel point load $8a$. If the point **I** is loaded the right reaction will be $1a$ and left the $7a$, etc.

Diagonals. To determine the maximum stress in **FG:** From what is shown above, the maximum stress in **FG** will come when panel points **4, 5, 6** and **7** are loaded. The left reaction due to **7** is a, to **6** is $2a$, to **5** is $3a$, to **4** is $4a$, making a total of $10a$. Cutting the truss as shown by the broken line, the left-hand part of the truss tends to move upward. **FG** must therefore be in tension, and must have in it a stress equal to $+10c$.

Verticals. To determine the stress in **JK,** cut the truss as shown by the inclined broken line. The maximum stress will then come when points **I, 2, 3,** and **4** are loaded, and the minimum when points **5, 6** and **7** are loaded. The maximum stress will therefore be $(1 + 2 + 3 + 4)a$ in compression or $-10a$. The minimum stress will be, reading from the left reaction, $(1 + 2 + 3)a = +6a$.

A uniform live load will bring in **JK** a load of $(-10 + 6)a = -4a$. It is evident that with the load distributed along the bottom chord, there would be no stress in **GH**; also that **AB** and **NO** will carry only the load hung at the panel point **I** or **7** respectively.

Counter Braces. Referring to Fig. 30, it has been shown that in **DE**, for instance, the maximum stress due to the live load will occur when the longer end of the truss is loaded, and will be positive. When, however, the short end only is loaded, the stress due to this live load will be negative. These members of the truss are ordinarily made so that they will carry only tension. If the positive load in **DE**, due to the dead load, is less than the negative load in **DE**, due to the live load, the part **DE** will be in compression, and the piece would have a tendency to buckle. To prevent this, a member crossing the same space in the opposite direction is provided. The resulting stress is taken on this additional member, which is called a counter brace, when by any combination of loading, the stress in this panel might bring compression in the diagonal.

When a counter brace is required in any panel, it is required in the corresponding panel at the opposite end, and in all the panels between, and, in any panel, that member is considered in service which has tension in it.

Suppose the truss shown in Fig. 30 weighed 80 tons. The maximum live load stress in **DE** will be $+15c$, and the minimum load would be $-3c$. The uniform would be $+12c$. As now the truss weighs 80 tons, $8a = 10$ tons, and the stress in **DE**, due to the dead load, is

$$(a) \quad \left(\frac{C}{A}\right)$$

$$(15-3)c = (15-3) \times \frac{10}{8} \times \frac{25}{20} = 18.75 \text{ tons.}$$

If $3c$ for the live load is greater numerically than 18.75 tons, a counter brace will be required in the section **DE**, or, as $3c$ is to be greater than 18.75 tons, suppose c to be 7 tons. a is then $\frac{20}{25}$ of 7, or 5.6 tons, and the panel point load $8a = 44.8$ tons. If, therefore, the truss has a live load of 44.8 tons per panel point, a counter brace would be required in the section **DE** to take the compression. The piece **EF** is made so that it can stand compression. As a counter brace is required in **DE**, it will also be required in **FG**, and also in the panels **HJ** and **KL**.

Initial Tension. It is often desirable to put a certain amount of initial tension in the counter braces. This initial tension does not affect any of the members outside of the rectangle in which the counter brace is located. Thus, in Fig. 31, initial tension of 4 tons, say, in the counter brace **DE** brings compression to the amount of $\frac{20}{25}$ of 4 ton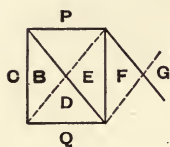s in **EF**, and to the amount of $\frac{15}{25}$ of 4 tons in **DQ**. This brings a tension of 4 tons in the diagonal **BD**, so that all the members in the rectangle are affected by the initial tension.

If initial tension was also put in the counter brace **FG**, the member **EF** of the original rectangle would have in it the compression due not only to the initial tension in **BD**, but also to the initial tension in **FG**. The effect of the initial tension is to delay the time at which the stress is taken from the main brace and carried by a counter brace, and it might be possible, by giving the counter brace sufficient initial tension, to insure that the load never is entirely taken off the main diagonal.

Fig. 31.—Initial Tension.

Wind Bracing. Every structure of this type is made up of two trusses at the right distance apart to form the proper platform for the moving load. These two trusses are tied together, so that the forces tending to overthrow the truss are properly carried to the piers supporting the trusses. The forces tending to overthrow a truss of this sort when loaded on the bottom chord are caused by the wind, and the bracing between the two trusses is called the wind bracing.

Fig. 32 shows the truss with the wind bracing on both top and bottom of the truss. Suppose the wind to act on the face of the truss in the direction as shown by the arrows. As shown in the right view, the tendency of the wind is to move the bridge bodily to the right, in addition to overturning it. These forces must be resisted by proper connections to the foundation. The top bracing (laterals), and the bottom bracing (laterals), constitute trusses of the same character as the main trusses themselves.

Verticals and Diagonals. As the only external forces that act on these trusses are those due to the wind, the wind is con-

sidered as acting in such a way as to bring the maximum
stress in the members. The wind is therefore assumed to
be of the same character as a live load, and to determine the
maximum stress in any member it is necessary to know the
maximum stress brought by a system of live loading. It has
been shown that when the longer portion of the span is loaded,
the maximum stress occurs in a diagonal or a vertical, and when
the short end of the truss is loaded the minimum stress occurs in
a diagonal or a vertical. To determine these stresses take, for
example, the panel **C′D′.** With the wind on the side as shown,
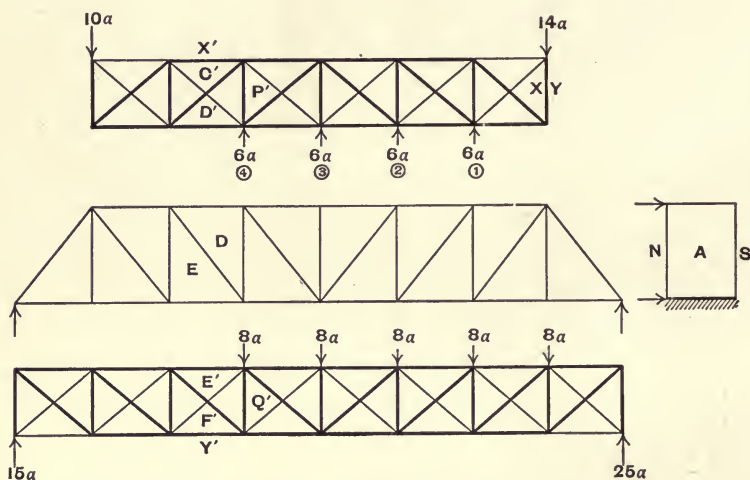
Fig. 32.—Wind Bracing.

called for convenience the north side, the maximum stress comes
in the diagonal in **C′D′,** when the load is on the panel points
1, 2, 3, and **4,** the load on the right panel point being carried
directly on **XY** and not on the truss. As this truss has six
panels, the panel point load is $6a$, and the reaction at the left is
$10a$, omitting the load at the end of the truss, which does not
affect the stress in the diagonals, and the right reaction is $14a$.
The portion of the truss in use is that shown by the heavy lines,
and the maximum stress in **C′D′** is then $+10c$. The maximum
stress in **D′P′** is $-10a$. The minimum stress in **C′D′** is zero and
in **D′P′** is zero.

Chord Stresses. To determine the maximum stress in **DD′**, note that the maximum chord stress occurs when the truss is entirely loaded. When this is the case, the top lateral bracing is as shown in Fig. 33, and the portion of the truss in use is shown by the heavy lines. The stress in the member **DD′** is

$$(15 \times 2 - 6 \times 1)b = -24b.$$

If the wind, instead of acting on the north side, acts on the south, the stress in the member **DD′** is the same as the stress in the member **C′X′**, with the wind on the north. The minimum stress in **DD′** is therefore $+(15 \times 1)b$.
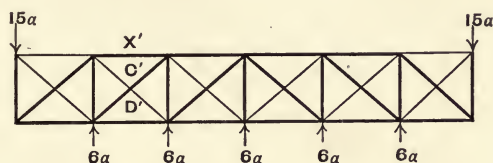
Fig. 33.—Chord Stresses.

To determine the maximum stresses in the bottom laterals, proceed in exactly the same way, but use $8a$ for the panel point load. The maximum stress in the diagonal **E′F′** is obtained by first marking off those portions of the truss which are in service, as is shown by the heavy lines at the bottom of Fig. 32, and the maximum stress in **E′F′** is $+15c$, and the minimum is zero. The maximum stress in the piece **E′Q′** is $-15a$ and the minimum is zero. In the same way as before, to determine the maximum and minimum stress in **EE′**, due to the wind load, we have a uniform load of $8a$ acting all over the truss. The reactions are then $28a$. The maximum stress in **EE′** is then $(28 \times 3 - 8 \times 2 - 8 \times 1)b = -60b$. The minimum stress in **EE′** is the same as the stress in **F′Y′**, with the wind on the north, or $(28 \times 2 - 8 \times 1)b = +48b$.

As a tentative figure in determining wind load, it is sufficient to measure in running feet the length of all members of the main truss, and assume that these members are each one foot wide. This will give us the number of square feet on which the wind would act on the entire truss, and, dividing by the total number of panel points, in this particular case, 16, will give us the panel point load. In addition, a panel point load equivalent to the area of the side of a train standing on the loaded

3

portion of a bridge is carried on the bottom chord, if the bridge
is a through bridge, or on the top chord, if the bridge is a deck
bridge. This load may be taken at 400 lbs. per running foot of
the live load. In ordinary uncovered railroad bridges, no account
is taken of snow load. If, however, a truss of this sort is covered,
account must be taken of it, in determining the maximum load
that comes on the bridge.

Maximum and Minimum. To determine the maximum and
minimum load in every member of the truss it is necessary to
determine independently the stress brought on each member by
the various systems of loading, and the greatest numerical com-
bination that can be made is the maximum stress, and the one
that is algebraically farthest from this is the minimum stress
that comes in the member. It must be remembered that the
dead load and the initial tension are always present and the
others must be considered as present or not, as they are of value
in increasing or decreasing the total load on the member. The
table herewith shows the loads which must be taken into con-
sideration in determining the stress in the various types of
members in a built-up truss of this kind.

Verticals ⎫ Dead, maximum live, minimum live, initial
Diagonals ⎭ tension.

Chords—Dead, uniform live, initial tension truss, wind
          north, wind south, and initial tension in laterals.

Verticals and ⎫
Diagonals of ⎬ Maximum live wind, initial tension.
lateral system ⎭

A Non-Uniform Moving Load. When the load moving over
a truss is a non-uniform load, a combination of graphical and
algebraic methods is probably the simplest method of determin-
ing the maximum stress occurring in a member of the truss.
There are a few elementary principles by which to locate the
load on the truss, and by the use of these principles that position
of the load which will give the maximum stress in the chord
members, or in the diagonals or verticals, can be determined.

Referring back to Fig. 30, it is evident that if a non-uniform
load rolls on the truss from the right, the maximum stress in

DE, for instance, will occur when the shear in the panel **DE** is a maximum, because the shear is the sum of all the loads to the left of the section, and, therefore, the greater the shear, the greater the stress in **DE.**

To get the stress in the member **EP,** moments must be taken about the point **3,** and this stress is evidently a maximum, when the moment of all the forces to the left of **3** is a maximum.

Similarly, to get the stress in **DQ,** moments must be taken about **2,** or, rather, the point over **2** on the upper chord, and when the moment of the forces to the left of this point is a maximum the stress in **DQ** will be a maximum.

(1) *The maximum bending moment under any load occurs when the center of the truss bisects the distance between the load in question and the c.g. of the entire load.*
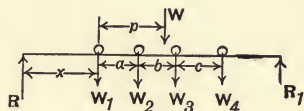
Fig. 34.—Maximum Bending Moment Under a Load.

Where should the second load W_2 in Fig. 34, be placed to give the maximum bending moment under it?

W is the total load and the c.g. of the load is a distance p from the left load. Call l the length of the truss. The reactions are

$$R_1 = \frac{W(p + x)}{l} \quad R = \frac{W}{l}(l - p - x).$$

The bending moment under W_2 is

$$R(x + a) - W_1a = \frac{W}{l}(l - p - x)(x + a) - W_1a.$$

This is a maximum when the first derivative is zero, or

$$d(l - p - x)(x + a) = o;$$

or, $\quad d(lx - px - x^2 + al - pa - ax) = l - p - 2x - a = o;$

or, $\qquad x = \dfrac{l - p - a}{2} \quad$ and $\quad x + a = \dfrac{l}{2} - \dfrac{p - a}{2}$

or, *the center of the truss bisects the distance between the c.g. of the load and the load in question.*

Example. Where should the loads, Fig. 35, be placed to give the maximum bending moment under each **W**? *c.g.* is 4.5 feet from the first load.
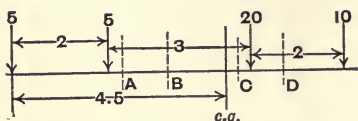
Fig. 35.—Example, Bending Moment.

When the center of the beam is at **A**, 2.25 feet from the left load, then the bending moment under the left load is a maximum.

When the center of the beam is at **B**, 1.25 feet from the second load, then the bending moment under the second load is a maximum.

When the center of the beam is at **C**, 0.25 feet from the 20-ton load, then the bending moment under the 20-ton load is a maximum.

When the center of the beam is at **D**, 1.25 feet from the right-hand load, then the bending moment under this load is a maximum.

(2) *To determine which of two loads gives the greater bending moment at any point.*

To determine whether in Fig. 36 the right load of R_1 or the left load of R_2 gives the maximum bending moment at a distance
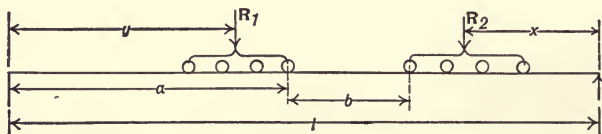
Fig. 36.—Maximum Bending Moment.

a from the left end of a beam whose length is l. With the loads as shown in the figure, the bending moment at a is

$$\frac{R_2x + R_1(l - y)}{l} a - R_1(a - y)$$

If the loads all move to the left a distance b, the bending moment at the point whose distance is a from the left end, is

$$\frac{R_2(x + b) + R_1(l - y + b)}{l} a - R_1(a - y + b).$$

Subtracting the first from the second, if

$$\frac{R_2 b + R_1 b}{l} a - R_1 b \text{ is } +, \text{ the second is the greater.}$$

Or, as $W = R_1 + R_2$,

If $\dfrac{a\,W}{l} - R_1$ *is* $+$, *the second is the greater.*

Example. To determine which of several loads if placed at $\dfrac{1}{4}$ the length of a girder or truss from the left end will give the greatest bending moment there, assuming the loads to be 3, 4, 8, 16, 20 and 20 tons.

$W = 71.$

$$\frac{a}{l} = \frac{1}{4}, \frac{a\,W}{l} = \frac{71}{4} = 17\frac{3}{4}.$$

$\left.\begin{array}{l} 17\dfrac{3}{4} - 3 - 4 - 8 = + \\[2ex] 17\dfrac{3}{4} - 3 - 4 - 8 - 16 = - \end{array}\right\}$ The 16 ton load at $\dfrac{1}{4}$ distance gives greatest bending moment.

(3) *To determine which of two loads gives the greater shear at any point.*

At a from the left, Fig. 37, the left reaction is the shear and is $\dfrac{Wx}{l}$, x being the distance from the right reaction to the center
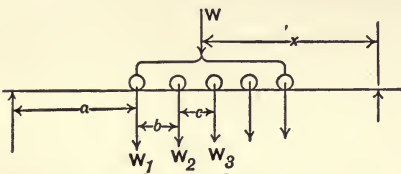
Fig. 37.—Maximum Shear.

of gravity of the loads and l the length of the beam.

When the loads move to the left a distance b, the shear at a is $\dfrac{W(x+b)}{l} - W_1$, and taking the difference,

if $\dfrac{Wb}{l} - W_1$ is $+$, the second shear is greater than the first. Now move the loads a distance c to the left. The shear at a becomes

$$\frac{W(x+b+c)}{l} - W_1 - W_2,$$

and this is greater than the last if $\dfrac{Wc}{l} - W_2$ is $+$.

Therefore, if $\dfrac{W}{l}$ *is greater than* $\dfrac{W_1}{b}$ *move the next load up.*

If $\dfrac{W}{l}$ *is greater than* $\dfrac{W_2}{c}$ *move the next load up, etc.*

Thus, if four loads roll over a 30-foot span, as shown in Fig. 38, $W = 55$, $\dfrac{W}{l} = 1.83$, $\dfrac{W_1}{b} = 1.66$, $\dfrac{W_2}{c} = \dfrac{5}{4} = 1.25$, $\dfrac{W_3}{d} = \dfrac{20}{8} =$

Fig. 38.—Example, Shear.

2.5. Consequently, the shear at any point is greater with the first 20-ton load at that point.
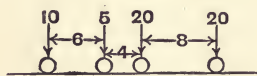
If the load, instead of being carried on the beam or truss directly, is carried as on the floor of a Pratt truss, the shear with the first load at a panel point is $\dfrac{Wx}{l}$. With the second load at this point, it is

$$\frac{W(x + b)}{l} - \frac{W_1 b}{\dfrac{l}{n}}$$

where $\dfrac{l}{n}$ is the length of one panel. And the second is greater than the first if $\dfrac{Wb}{l} - \dfrac{W_1 nb}{l}$ is $+$, or if $\dfrac{W}{n} - W_1$ is positive. If the next load is moved to the panel point, the shear becomes

$$\frac{W(x + b + c)}{l} - \frac{W_1(b + c)}{\dfrac{l}{n}} - \frac{W_2 c}{\dfrac{l}{n}}$$

and this is greater than the preceding if

$$\frac{Wc}{l} - \frac{W_1 nc}{l} - \frac{W_2 nc}{l} \text{ is } + \text{ or if } \frac{W}{n} - W_1 - W_2 \text{ is } +.$$

Therefore, if $\dfrac{W}{n} - W_1$ *is* $+$ *move the next load up for the greater shear.*

Thus, if on a 10 panel 160-foot span, we have loads as shown in Fig. 39.

$$\frac{W}{n} = \frac{80}{10} = 8.$$ If the load is moving to the left, the greatest shear occurs to the left of the panel point, when the 16-ton load is on the panel point. If the load is moving to the right the greatest shear occurs with the first 30-ton load on the panel point.
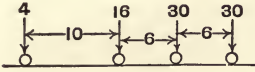
Fig. 39.—Example, Shear.

Chord Stress from the Equilibrium Polygon. As the equilibrium polygon is a bending moment diagram, and as the bending moment at any point is the height of the equilibrium polygon times the pole distance, if the pole distance is made the height of the given truss, then the height of the equilibrium polygon, properly drawn, will represent the stress in the chords. If an equilibrium polygon is drawn for the loads, and by the use of the
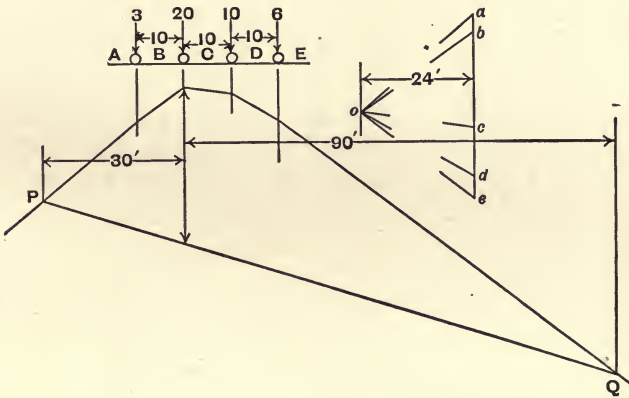
Fig. 40.—Equilibrium Polygon for Chord Stresses.

preceding principle, the load which will give the maximum bending moment at any point is determined, the corresponding stress in the chord can be immediately read off from the bending moment diagram.

Thus, in Fig. 40, a series of loads are given, **AB, BC, CD, DE,** acting at the distances apart as shown. Draw the force polygon,

abcde. With the pole distance equal to the height of the truss, mark off any point *o*, and draw the lines *oa, ob,* etc. Through the space **B** draw a line parallel to *ob.* Crossing the space **A** draw the indefinite line parallel to *oa;* through the space **C** draw a line parallel to *oc,* and similarly through **D** and **E** draw corresponding lines, thus determining the portion of the equilibrium polygon, terminating under each of the loads. Suppose that the maximum bending moment at a point $\frac{1}{4}$ the length of the truss from the left-hand end exists when the second load is at that point, this having been determined as already shown. Lay off on either side of the line of action of **BC**, $\frac{1}{4}$ the length of the truss to the left, and $\frac{3}{4}$ to the right. The closing line of the equilibrium polygon, therefore, passes through the points **P** and **Q**. The maximum stress in the chord is the distance shown by the arrows on the line **BC**, and this distance, multiplied by the scale of the load, is the maximum stress in the chord.

Maximum Shear from Equilibrium Polygon. It has already been shown how to determine which of a series of loads must be at the panel point to cause the maximum shear at that point. The shear at any point is the left reaction minus the loads up to the point. As the left reaction is the moment of all the forces on the beam taken about the right reaction, and divided by the length of the beam, and as the moment of these forces is the height of the equilibrium polygon times the pole distance, it is evident that the left reaction is the height of the equilibrium polygon, if the pole distance is the length of the truss. It is therefore possible to take the shear from the equilibrium polygon directly, if the equilibrium polygon is drawn with the length of the truss as a pole distance.

Suppose it is desired to determine the maximum shear at the point $\frac{1}{4}$ the length of the truss from the left-hand end, with the series of loads as shown in Fig. 41.

Lay off the force polygon *abcde.* With a pole distance equal to the length of the truss, draw the lines *oa, ob, oc, od* and

oe, and draw the corresponding portions of the equilibrium
polygon *a'b'c'd'* as shown. Now suppose it has been determined
that placing the load **BC** at the given point will give the maximum
shear. Placing the point of the beam, at which the maximum
shear is required at **BC** and laying off the beam as shown from
X to **Y**, the moment of all the forces on the beam is represented
by the distance **PQ**, and as the force **AB** is between the point at
which the maximum shear is being measured and the left-hand
end, we must subtract the force **AB** from **PQ**, leaving a distance
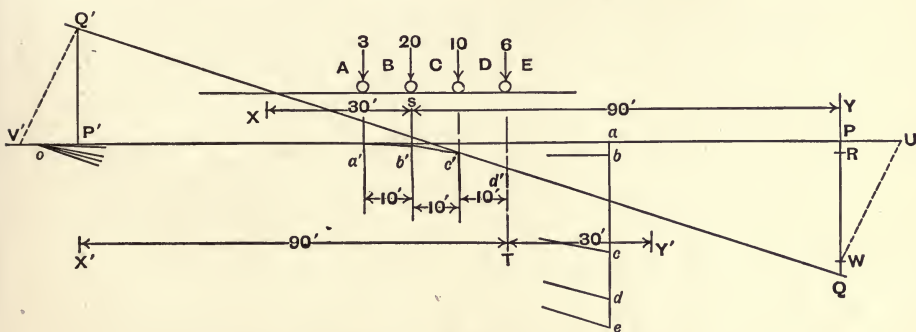
Fig. 41.—Shear from Equilibrium Polygon.

RQ for the shear in the space **B**, and this is the maximum value
of the shear that can occur at the point taken, if the loads roll
over the beam. When the loads are not symmetrical, it is neces-
sary to determine the maximum shear due to the loads running
either way.

The maximum shear occurs under the second load when
moving to the left, but when the load is turned around in this
case it occurs under the first load. It is then necessary to mark
off a second position of the truss, having the point **S** under the
load **DE**, at the point **T**, the truss occupying the position **X'Y'**
and the distance **Y'T** being the same as the distance **XS**. The
height of the equilibrium polygon is then the distance **P'Q'**, and
the maximum shear comes at the point **S**, with the second load
BC over **S** if **RQ** is greater than **P'Q'**. If, however, **RQ** is less
than **P'Q'**, the maximum shear at **S** will come when the loads
are reversed, and when the load **DE** is over the point **S**, and the
amount of this shear is **P'Q'**.

Diagonals. If the stress in the diagonal is required, and if the maximum shear occurs under **DE**, draw a line **Q′V′** at the angle that the diagonal makes with the vertical and this is the stress in the diagonal. If, however, the maximum shear occurs under **BC**, and **AB** is on the panel in which the diagonal stress is desired, that portion of **AB** which would be carried on the left panel point of the section must be taken from **PQ**. That is, the amount **QW** is the part of the force **AB** taken from the shear, and is found by multiplying **AB** by the distance between **AB** and **BC**, and dividing by the length of the panel. The diagonal stress, or the stress in the diagonal, instead of being **Q′V′** is **WU**

Deflection of a Truss. When a load is slowly applied at any point in a truss, the truss will deflect, and the amount of work done by the load is one-half the load times the amount of the deflection. This quantity of work is stored up in the individual members of the truss, and each one of them takes its share of the work. Call the total stress in any member **P** lbs., **A** its area, **E** the modulus of elasticity of the material, and **L** the length. The extension or shortening of the piece under the stress **P** is $\dfrac{P \times L}{A \times E}$; the amount of work stored up in the piece is one-half **P** times the extension, or is $\dfrac{1}{2} \dfrac{P^2 L}{A E}$.

If this quantity is determined for each member of the truss, and the total value determined, the total must be equal to the work done by the external load, or, calling **W** the external load and f the deflection, this quantity must equal $\dfrac{Wf}{2}$ or

$$\frac{Wf}{2} = \Sigma \frac{1}{2} \frac{P^2 L}{A E}.$$

If the deflection is required at some other point than the point at which the load **W** is applied, place a fictitious load at the point at which the deflection is desired, and determine the stresses **P′** in each member of the truss. The deflection produced by the actual loading due to the actual change in length of any one member, will be greater or less than this elongation in the proportion that **P′** is greater or less than **W′**. It is therefore possible to calculate the amount of deflection $f′$, due to the actual stress in

each piece, and, by adding these partial deflections, to get the total deflection at the point desired in the truss.

If the load **W′** is placed at the point at which the deflection is desired and **P′** is the stress in any one member, then that part of the deflection due to the lengthening or shortening of that one member is, from the above equation,

$$f' = \frac{P'}{W'} \frac{P'L}{A E}$$

or, the deflection at this point due to **W′** is to the increase (or decrease) in length of the piece as **P′** is to **W′**. That is, any increase in length of a member affects the deflection at any point in the ratio of **P′** to **W′**. Now, under a load **P**, the increase in length of the member is $\frac{PL}{AE}$, the deflection at the required point, due to the increase in length in this piece, is therefore $\frac{P'}{W'} \frac{PL}{AE}$. If this value is determined for each member of the truss and the algebraic sum is taken, this sum is the total deflection at that point. If in any member **P** and **P′** have the same sign, the partial deflection is +, and if opposite signs, the partial deflection is −.

Referring to Fig. 42, suppose the truss is as shown with the external loads as indicated on the upper chord of the frame, and it is required to find the deflection of the corner **ABCK**. First draw the upper force polygon, having the forces **FG, GH, HJ, JK** and **KF** acting, and tabulate the stresses acting in the various members, as shown in column 2 of Table 1. Assume, now, that only a load **KK′** of, say, 10,000 lbs. is acting at the corner **ABCKK′** and draw the lower force polygon as though the load **KK′** only were acting, and tabulate the stresses exerted by this load in every member of the truss, as shown in column 3 of the table. In the sixth column is tabulated the value $\frac{PL}{AE}$, and the last column gives this value in the sixth column times $\frac{P'}{10000}$. The algebraic sum of these values gives the total deflection at the point **ABCKK′**. If it is desired to know the distance any point

of the truss has moved in any direction, it is only necessary to put an auxiliary force **KK′** so that it will act at the point and in
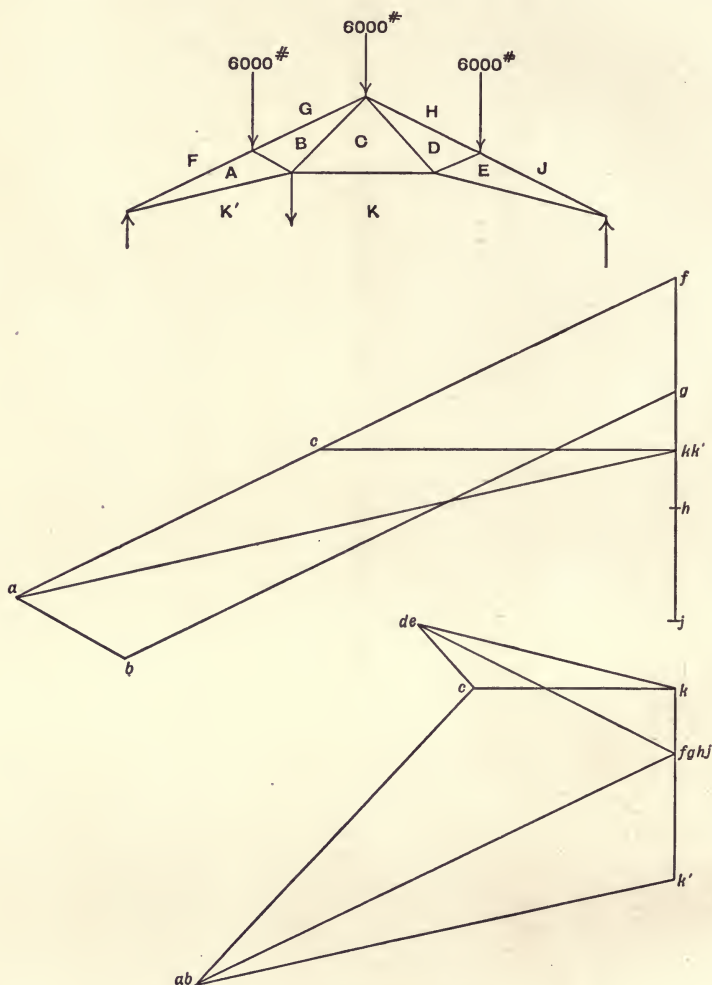

Fig. 42.—Deflection of a Truss.

the direction desired, and the deflection determined as above will be the deflection in the desired direction.

Graphics of Machines. When a force F is applied to one part of a machine at the receiving end, and moves a distance s

in a given time, the quantity of work received by the machine is $F \times s$. At the other end of the machine, another force P is exerted by the machine, and the distance s' through which this P moves, while the force F is moving through the distance s, is such that the quantity of work delivered by the machine is less than the amount of work received by the machine, the difference being the amount of work that is used up in friction, it being assumed that all parts of the machine are moving at the same velocity at the beginning and the end of the time taken.

	P	P′	A Area □″	L Length ft.	$\dfrac{PL}{AE}$	Deflection at K K′
FA	− 40200	− 30000	7.22	13.87	.0309	+ 0927
GB	− 33000	− 30000	7.22	13.87	.0253	+ 0759
HD	− 33000	− 17000	7.22	13.87	.0253	+ 0430
JE	− 40200	− 17000	7.22	13.87	.0309	+ 0525
KE	+ 36900	+ 15900	3.86	17.46	.0668	+ 1062
KC	+ 18750	+ 11500	2.12	16.00	.0566	+ 0651
KA	+ 36900	+ 27750	3.86	17.46	.0668	+ 1851
AB	− 6900	O	2.38	4.92	.0057	———
BC	+ 15600	+ 22500	1.62	11.31	.0436	+ 0981
CD	+ 15600	+ 5600	1.62	11.31	.0436	+ 0244
DE	− 6900	O	2.38	4.92	.0057	———
					Deflection 0″.7430	

Table 1.—Deflection.

Efficiency. If it is assumed that the force P moves through a distance s', $F \times s$ is equal to $P \times s'$ if there is no friction, and $F \times s$ is greater than $P' \times s'$, if there is friction. The efficiency of the machine is ordinarily understood to be the ratio of P′ to P in the two quantities written above, as the mechanism makes it necessary that the distance s' is the same with friction and without. *The effect of friction in a machine is always to reduce the quantity of work that may be done by the machine if a given force acts on the machine.*

Friction. By the coefficient of friction is meant the force exerted parallel to the rubbing surfaces divided by the force exerted at right angles to the rubbing surfaces.

By the friction angle is meant the angle whose tangent is the coefficient of friction

In Fig. 43, is shown a block, acted on by a force **F**, and having a weight **W** normal to the rubbing surface *ab*. If there is no friction between the surfaces, the block **A** will move with acceleration, and cannot be in equilibrium.

Fig. 43.—Friction.

If there is friction between the surfaces, as the force **F** gradually increases from zero, the block **A** will remain at rest until the friction force exerted between **A** and the plane *ab* is just equal to the force **F**. When this is the case, the reaction of the plane *ab*, instead of being vertical as shown by the broken line, takes the position shown by the full line **R**, the angle between the vertical and the inclined line being the friction angle.
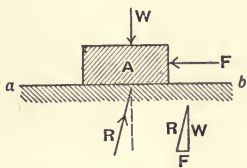
The resistance of the plane *ab* to the motion is such that the vertical component of the force **R** balances the weight **W**, and the horizontal component balances the force **F**, or, as shown in the small triangle, the forces **W**, **F** and **R** represent the triangle of forces that are exerted on the block **A**. As the block **A** is a three-force piece, these three forces must pass through the same point. The effect of friction on a block of this sort is to distribute the pressure between the block **A** and the surface *ab* unevenly.

If the force **F** acts as shown in the figure, then the resistance **R** cannot act through the center of the lower face of the block **A**, but must act through a point to the left of the center. Therefore, the pressure between the block **A** and the piece *ab* must be greater on the left-hand side of the block than on the right-hand side, as shown in the figure. The total amount of the force is not changed by the change in the distribution of the application of the forces.

Wedge Friction. Suppose now instead of having a flat block, as shown in Fig. 43, the block is wedge-shaped as shown at **A**, in Fig. 44, and suppose that the load **W** is carried on the

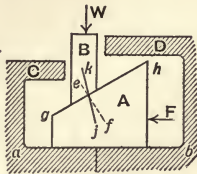block **B**, and that the pieces **C** and **D** are attached to the plane *ab*, on which the wedge **A** slides. Suppose the forces are just sufficient to hold the block **A** in the position shown without moving. The wedge **B** will tend to slide down the inclined surface of **A**, and will be prevented from sliding, if at all, by the friction between these two surfaces. The normal force between **A** and **B** acts in the direction *ef* normal to the wedge surface *gh;* and if the tendency is to slide down the wedge surface *gh*, the friction force between **A** and **B** acts along the friction surface *gh*, and, as far as the face of **B** is concerned, acts up the inclined surface.

Fig. 44.—Wedge Friction.

If the line *jk* is drawn so that the angle between the normal *ef* and *jk* is the friction angle, then the force acting in *jk* is the resultant force exerted between **A** and **B** when **B** is moving over **A**. If this has a component to the left, then **B** will slide down the wedge surface *gh* until it comes in contact with the stop **C**. If it has a component to the right, then the block **B** will not move on the block **A**. When it is just on the verge of moving, the line *jk* will be vertical and there will be no component in either direction, that is, the friction angle between *ef* and *jk* will be such that *jk* is vertical, and this angle will be equal to the angle between *gh* and the horizontal line *ab*, which is the angle of the wedge.

One can therefore tell whether the block **B**, if free, will slide down, or have to be forced down, if the magnitude of the friction angle is known.
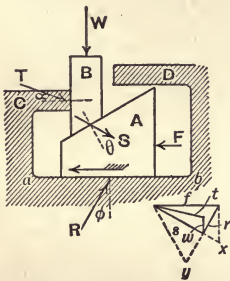
Suppose it is assumed that the friction angle is such that **B** will tend to slide to the left, and suppose it has been allowed to come in contact with the stop **C** as shown in Fig. 45. Suppose the force **F** acts, and the block **A** moves to the left, and it is desired to know how much force at **W** can be raised by the force **F**. The piece **A** is a three-force piece. The forces acting on it are 1st, the force **F**; and 2d, as the movement of **A** is to the left,

Fig. 45.—Moving Wedge.

a force in the direction of **R**, making a friction angle ϕ with the vertical; and, 3d, a force **S** making the friction angle θ with the normal to the wedge surface. As these are all the forces that act upon **A**, a force polygon can be drawn, having f, r and s as sides, thus determining the value of the forces acting on the wedge **A**.

The forces acting on the wedge **B** are, 1st, the reaction due to **S**; 2d, the weight **W**; and 3d, the force **T** exerted by the piece **C**. As the wedge **B** tends to move upward, the force exerted by **C** tends to hold it down, and this force makes an angle a with the normal to the surfaces. The piece **B** is therefore a three-force piece. The value of **S** is known and the value of **T** and **W** can be found by completing the force triangle *stw*. Had there been no friction at **R**, the force s would have extended to x, and therefore **W** would have been larger. Had there been no friction at **S**, the force **S** would have extended to y, larger than s, and again **W** would have been larger.

As it will be necessary to make a distinction between the direction in which the forces act and the direction in which the pieces move, an arrow head standing alone will represent the direction of the force, while the feathered arrow will represent the direction of movement.

Suppose, in Fig. 45, that the piece **A** had been moving in the opposite direction, and that the friction on the wedge was not sufficient to cause the piece **B** to move with the piece **A**. Fig. 46, then, will show the direction in which the forces act on the pieces. The direction of **R** is reversed to the opposite side of the normal; the direction of **S** is on the opposite side of the normal, but it is still to the left of the vertical. As the piece is sliding down along the stop **C**, the
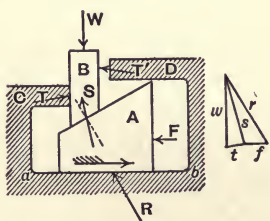
Fig. 46.—Wedge Friction.

friction force acts in the direction **T** as shown. Having given, then, the weight **W**, the triangle for the forces acting on **B** can be drawn, determining the amount of the force **S**, and, drawing through the extremities of s lines parallel to **R** and **F** will determine the value of all the forces acting. In this particular

case, as the force *s* acts downward on **A** from the piece **B**, the force **F** must act in the opposite direction to that shown by the arrow. The friction might have been such that *s* would fall on the other side of *w* in the force polygon, in which case the force **T** would act as shown by the arrow **T′** in Fig. 46. The piece **B** would have been brought up against the stop **D**, instead of the stop **C**, the other force remaining in the direction shown.
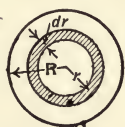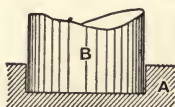
It is not at all necessary in cases of this sort to know the point of application of the forces, as the amounts do not depend on the point of application, but on the direction, there being only three forces acting on each piece.

Friction on a Pivot. In Fig. 47, a step is shown at **A**, in which turns, without touching the sides, the bottom of the vertical shaft **B**. It is assumed that the

Fig. 47.—Pivot Friction.

weight of the shaft **B**, or of **B** and the loads that it carries, is uniformly distributed over the entire bottom face of the shaft. Then, if **W** is the load, and **R** the radius of the shaft, the pressure per square inch exerted between the shaft and the footstep is $\dfrac{\mathbf{W}}{\pi\,\mathbf{R}^2}$. As the shaft **B** turns in the footstep **A**, the friction between the bottom of the shaft and the step tends to resist the motion, and on each square inch there is a friction force represented by the pressure per square inch times the coefficient of friction, or, calling *f* the coefficient of friction, on any square inch of the surface there is a force tending to resist the turning of the shaft, the amount of which is $\dfrac{\mathbf{W}f}{\pi\,\mathbf{R}^2}$. Evidently, to turn the shaft against the friction force, a turning moment must be put on the shaft, and the amount of this moment must be sufficient to overcome or balance the resisting moment due to the friction.

Take any narrow strip on the base of the shaft, and call the radius of the strip *r*. The length of the strip, then, is $2\pi r$, and, calling the width of the strip *dr*, the area is $2\pi r\,dr$. Now, on this strip is exerted a friction force represented by $\dfrac{\mathbf{W}f}{\pi\,\mathbf{R}^2}$ per square

inch, and the total friction force exerted on this ring is then $2\pi r dr\ \dfrac{Wf}{\pi R^2}$, and the moment of the friction force is $2\pi r^2 dr\ \dfrac{Wf}{\pi R^2}$.

The total amount of the friction moment is the summation of this quantity between the radii R and zero, or the friction moment is

$$\int_0^R 2\pi r^2 dr\ \frac{Wf}{\pi R^2} = \frac{2WfR}{3};$$ or, if it is assumed that the friction force

is produced by two equal loads, each one half of the total load, the friction moment is produced by $\dfrac{Wf}{2}$, acting at $\dfrac{4}{3}$ of the radius.
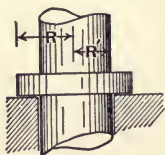
Fig. 48.—Collar Bearing.

If, instead of a vertical shaft being carried on a step-bearing, as shown in Fig. 47, it is carried on a collar, as shown in Fig. 48, the pressure per square inch on the collar, calling R the greater radius and R' the smaller radius, is $\dfrac{W}{\pi(R^2 - R'^2)}$ and the friction force per square inch is f times this, or

$\dfrac{Wf}{\pi(R^2 - R'^2)}$. The moment of the friction forces acting on any ring of the collar having a radius r is $2\pi r^2 dr\ \dfrac{Wf}{\pi(R^2 - R'^2)}$, and for the total friction moment we have the integral,

$$\int_{R^1}^R 2\pi r^2 dr\ \frac{Wf}{\pi(R^2 - R'^2)}, \ \text{or}\ \frac{2}{3}\frac{(R^3 - R'^3)}{(R^2 - R'^2)}\ Wf.$$

Screw Friction. The force exerted by a screw is partially the force exerted between two wedge surfaces, and partially that between a collar and its bearing. From the formula deduced in the last article, it is easy to determine that for a collar of reasonable dimensions it can be assumed that one half the load on the collar is exerted at the center of the width of the face. Thus with a collar having $2\frac{1}{2}''$ outside diameter, and $2''$ inside diameter, the center of the face has a diameter of $2\frac{1}{4}''$, and it can be assumed that half the load on the collar is carried on one side

of the center, and half on the other side of the center, at a distance

from the first one of $2\dfrac{1}{4}''$. Following the formula above deduced,

the exact value is $\dfrac{61}{27}'' = 2\dfrac{7}{27}''$ instead of $2\dfrac{1}{4}''$. When we are

dealing with screws, therefore, it can be assumed that half the
load on a screw is carried on a screw surface at half the width of
the surface from the outside of the screw, and it is then the
equivalent of a wedge, the angle of which is the angle of the
screw at this point.

To determine the angle of the screw, the radius at which
the load is supposed to act is determined as stated above.
The circumference of a circle of this radius is the length of the
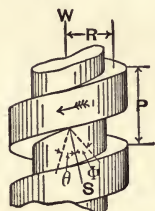
Fig. 49.
Screw Friction.

base of the wedge, while the height of the wedge
is the pitch, and therefore the screw may be
treated as simply a special case of the wedge.
Thus, in Fig. 49, suppose the load **W** acts down-
wards on the screw as shown, the nut not being
indicated, and assume that the screw is turning
clockwise when looking down on it. It is then
screwing down into the nut. Assume that the
load **W** is carried half on either side of the screw
at the center of its face. The distance **R**, as
shown in the figure, is the radius of the screw, and the distance
between the successive threads, marked **P,** is the pitch of the
screw. Assume that the load is concentrated on a portion of
the thread immediately in front and a corresponding portion

Fig. 50.
Screw Forces

diametrically back of it. The force exerted by
the nut on the face of the screw and acting
normal to the screw surface, is as shown by the
full line at **S**, and acting at the corresponding
angle with the vertical on the opposite side, for
that part of the screw surface that is away
from us. If we draw a vertical line, Fig. 50,
whose length is **W**, and through the upper
point draw a line parallel to **S**, and at the center of **W** draw a
horizontal line, this horizontal distance is the force f which
must be exerted at each end of the radius **R** to prevent the
load **W** from forcing the screw down, if there is no friction. If
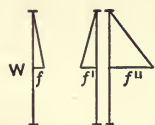
there is friction, the direction of motion being as shown, the
friction tends to resist the movement of the thread in the nut
and the friction angle θ should be laid off in the direction shown
in Fig. 49. The force necessary to overcome the friction, and
cause the screw to turn downward in the nut, is represented by
the distance f' in Fig. 50. If, instead of turning the bolt in the
direction of the arrow, it is caused to turn in the opposite direc-
tion, that is to unscrew it from the nut, the friction force
should be laid off in the opposite direction, as shown at ϕ, and
the force now required, acting at a distance **R**, to raise the load
W, is the force f'', the f in each case being laid off at the center of
the load **W**.

It is perhaps worth noting that, starting with no friction,
a twisting moment must be exerted on **W** to prevent it from
running down. As the coefficient of friction increases from
zero, the twisting moment that must be exerted, as, for instance,
by means of a wrench, becomes less and less, until, when the
friction angle and the angle of the screw are the same, the load
W would not run down of itself. If the friction is still greater,
the force must be exerted by the wrench in the opposite direc-
tion to cause the load to travel down.

Wedge and Screw. Suppose the apparatus to be as shown
in Fig. 51,—To determine the force that must be exerted at the
ends of the handle **A–A** to raise the weight **W**, with and with-

Fig. 51.—Screw and Wedge.
No Friction.

out friction. The full lines in Fig.
51 represent the lines of action of the
various forces acting between the dif-
ferent pieces when there is no friction.
The lines are all drawn normal to the
various surfaces, and the force poly-
gon can be drawn as shown below.
To determine the direction of the line
E, first draw a line lm at right angles
to the axis of the screw, of a length
equal to 2π times the radius to the
center of the screw surface. From l
lay off the distance ln equal to the pitch of the screw. The
line mn then represents the line of the screw surface, and
the line **E** is drawn at right angles to mn. To determine the

force required at **A**, multiply the force f by the radius of the circle from which lm was determined, and divide it by half the distance between **AA**, and the quotient is the force that must be exerted at the points **A** and **A**, to raise the weight **W** without friction. In the figure, it must be exerted toward the observer at the bottom, and away from him at the top.

With Friction. If there is friction at all the surfaces, the forces acting are as in Fig. 52, each force being laid off in the proper direction showing the line of action of the forces with friction.

Lay off the force **W** as before. Lay off the various forces in the force polygon as shown. To determine the force required at **A**, we have friction now both on the screw and on the collar. f represents the force acting at the center on one side of the screw, tending to resist the turning of the screw, and k represents the force on one side of the collar tending to resist the turning of the collar, and both of these must be overcome by the force acting at **A**. Therefore, the force required at **A** is the force f times the radius of the screw plus the force k times the radius of the collar, divided by the radius on which **A** acts. The radius of the screw and the radius of the collar referred to above is the radius to the center of the acting face.

Fig. 52.—Screw and Wedge. With Friction.

Bearing Friction. When a piece turns in a bearing, the conditions are those shown in Fig. 53. If there is no friction the point of contact between the bearing and the shaft is at the center of the bottom of the bearing, and the weight acting downward through the center of the shaft is exactly balanced by the force acting upward through the center of the

Fig. 53.—Bearing Friction.

bearing. As there is assumed to be no friction between the shaft and the bearing, if the shaft is caused to rotate, the point of contact will remain directly below the center of the shaft.

Whenever there is friction and the shaft is turned in the direction of the arrow, the shaft rolls up the side of the bearing toward the right into the position shown, the clearance between the bearing and the pin being very much exaggerated in the figure.

When the journal has reached as high a point as possible in the box, the point of tangency between the journal and the box will be at such a point that the tangent to the surface *ab* makes the friction angle with the horizontal *ac*, as the bearing is exactly the case of rolling a piece up an inclined plane or wedge having varying angles of inclination.

The force exerted between the bearing and the journal is made up of two parts, one of which is normal to the surface, and passes through the center of the journal, and the other is the friction force which is parallel to *ab*. As the journal is sliding down along the surface of the box, the resultant of these two forces is the vertical force *df* acting through the point of contact. It is vertical for the reason that if it inclined to the left, the shaft would have a tendency to move to the left down the slope on the box, and if it inclined to the right, the shaft would have a tendency to move further up the box, and as this is the permanent position of the journal, this line must be vertical, having no tendency to move the shaft sideways.

The forces acting on the journal are, then, the weight **W**, acting downwards through the center, and its equal **W**, acting upwards through *d* which supports the weight. As the two **W**s are not in the same line, they produce a turning couple in the opposite direction to the movement, and this turning couple must be balanced by the couple on the outside of the bearing, which causes the shaft to turn. As *de* in the figure is the direction of the normal force, the force *ef* is parallel to the direction of the friction force, and the angle *bac* or *edf* is the friction angle. The friction, therefore, causes the point of contact to move, in this case to the right, through a distance equal to *ex* or equal to the radius of the journal times the sine of the friction angle. Evidently, if the journal was turning in the opposite direction, the point of contact would be as far to the left as it is now to the right of the center of the shaft.

If, therefore, with the center of the journal *e* as a center,

a small circle is drawn having a radius equal to $r \sin \theta$, r being the radius of the pin and θ the friction angle, the line of action of the reaction **W** is displaced to the right or to the left, so that it becomes tangent to this circle. The circle having a radius of $r \sin \theta$ is called the friction circle.

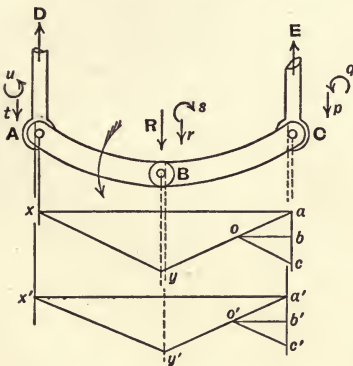**Bell Crank.** Having the bell crank lever, **ABC**, Fig. 54, with rods attached at **A** and **C**, the forces in these rods acting always vertically upwards, and the bearing at **B** holding the bell crank lever in place, to determine the effect of friction on the pins at **A**, **B** and **C**, and the relation between the forces exerted at **A** and **C**. The force in the piece **D** acts upwards as shown by the arrow, and in **E** acts upwards as shown by the arrow; therefore the bearing at **B** must exert a downward force on the pin at **B**. Suppose the point **A** is moving downwards, which is the direction of motion that is shown in the figure. Then, the force **E** is driving and the force **D** is the force that is being overcome.

Fig. 54.—Bell Crank Lever.

If there is no friction at either **A**, **B** or **C**, the relations between the forces acting at these points can be determined from the equilibrium polygon shown below at *axy*, the lines being drawn through the center of the pins, these lines being the lines of action of the forces **E**, **D** and **R**.

Laying off the distance *ab* equal to the force **E**, and drawing *bo* parallel to *ax* gives us *o* for the pole from which the equilibrium polygon is drawn. Drawing a line from *o* to *c* parallel to *xy* determines *bc* for the amount of the force **D**, if there is no friction.

If there is friction at the points **A**, **B** and **C**, the line of action of these forces will move parallel to themselves and in each case a distance equal to $r \sin \theta$, the r in each case being the radius of the particular pin talked about, and θ the friction angle for the materials at that particular place.

As the effect of friction is to reduce the effectiveness of the acting force, it is evident that the putting of friction at the center **C** has the same effect on the force **D** as though **E** acted on a shorter arm, that is, without referring to Fig. 53, it is evident that **E** will act on the left side of the friction circle.   In the same way, if there is friction at **B**, the force acting at **B** will move to that side of the friction circle which will make **E** less effective, and therefore the force at **B** will move to the right side of the friction circle.   If there is friction at **D** only, the line of action of the force **D** will move to that side of the friction circle which will make **E** less effective, that is, in this case, it will move to the left side of the friction circle, and the line of action of the forces will be as shown by the broken lines.

Again draw the equilibrium polygon by drawing any triangle *a'*, *x'*, *y'*.   Lay off *a'b'* the same length as the force *ab*, draw *b'o'* parallel to *a'x'* and through *o'* draw *o'c'* parallel to *x'y'*. This determines *b'c'* for the force exerted at **D**, and it is evidently less than the force exerted at **D** when there is no friction.

Use of Friction Circle.   While it is easy in this case to determine to which side of the friction circle the line of action of the force moves, it is not always so.   Fig. 54 can be used to illustrate what might be called a general method of determining the direction in which the force acts when the friction is taken into account. Take, for instance, the piece **E**.   The external force at the joint **C** acts downwards on the piece **E**, as shown by the small arrow *p*.   The angle between **E** and the arm **BC** is opening, and therefore the friction force acting on **E** tends to move **E** against the hands of a clock as shown by the small circular arrow *q* close by. Now, the line of action of the force moves to that side of the friction circle on which the arrow *p* and the arrow *q* follow each other, that is, **E** moves to the left side of the friction circle.

Take now, the joint **B**.   The external force **R** on the piece **ABC** at **B** acts downwards, as shown by the arrow *r*.   As the piece is turning against the hands of a clock, the friction force on the piece **ABC** acts as shown by the curved arrow *s*, and the line of action of the force at **B** moves to that side of the friction circle on which *r* and *s* follow each other.

Referring to the joint at **A**, the external force on the piece **D** at **A** acts downwards, as shown by *t*.   The angle between **D**

and **AB** is closing. Therefore, the friction force tends to hold this angle open, and the friction force then acts on the piece **D** in the direction as shown by the circular arrow *u*, and the line of action of the force **D** moves to that side of the friction circle, in this case the left side, where the arrows *t* and *u* follow each other.

If this method is mastered, little difficulty will be found in putting in the line of action of any force on any mechanism having friction at the bearings.

Horizontal Engine. Let Fig. 55 represent the piston rod, cross head, connecting rod and crank of an ordinary horizontal engine, the work done by the engine being the hoisting of a load, as shown at **W**. What force **F**, is required, with and without friction, to raise the load **W**? Without friction, the connecting rod **AB** is a two-force piece; the cross head **A** is a three-force piece, acted on by the force **S** from the connecting rod, the pressure **R** from the guides and by **F**. The crank, its shaft and the hoisting wheel form together a three-force piece, the acting forces being the connecting-rod force **S**, the load **W** to be raised and the resistance **T** of the bearings. Without friction, the center line shows the line of action of the forces which act on the connecting-rod passing through the center of the bearings. The force **R** from the guides acts through the point in which the line through the center of the connecting rod and the line of the force **F** intersect. The force **S** in the connecting rod, the line of action of the force **W**, and the force **T** acting on the bearings of the shaft pass through one point, and we have data enough to draw the force polygon as shown in the small figure.
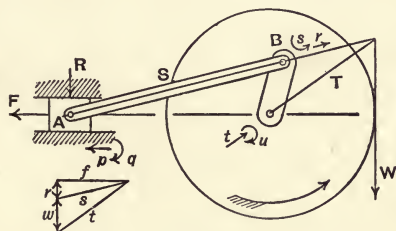
With friction at the bearings, it is first necessary to get the line of action of the force **S** acting in the connecting rod. If it had so happened that the crank and the center line of the engine were exactly at right angles, at the cross-head end of the connecting rod the force would have passed through the

center of the pin, while the force at the right-hand end of the connecting rod would pass either above or below the center, and would be tangent to the friction circle.

To determine whether it passes above or below, assuming that the angle between the crank and the connecting rod is as shown, the external force at the lower left-hand end of the connecting rod acts in the direction as shown by the small arrow p, Fig. 55. The connecting rod is still moving up, and therefore the friction force acts as shown by the small arrow q, and the force then comes to the lower side of the friction circle. At the end B of the connecting rod, the external force acts on the connecting rod as shown by the small arrow r. The connecting rod is moving so that the angle between the connecting rod and the crank is opening. The friction force, therefore, acts in such a way as to close the angle between the two, and therefore acts on the rod in the direction of the small circular arrow s and the line of action of the force in the connecting rod then comes to the lower side of the friction circle, at this end.

The external force T acting on the crank, acts in the direction of the small arrow t. As the crank is moving against the clock, the friction force at that point acts as shown by the small arrow u, and therefore the line of action of the force at the shaft moves to the left side of the friction circle. The diagram of forces may now be drawn.

In Fig. 56, is reproduced the machine with the lines of action of the forces that act on the parts with friction, only enough being shown to draw the lines of action of the forces.

Fig. 56.—Forces with Friction.

Starting with F, the forces f', r', s', t' and w' are obtained and the efficiency of the mechanism for this position is the ratio of the w' from this figure to the w from the preceding one. Starting with the same F there results a lesser load W', because of the friction, or if it is required to lift the same W, a greater F will be necessary with friction than without it.

Grasshopper Engine. As another example, suppose the mechanism to be as shown in Fig. 57, in which **A, B,** and **C** represent fixed points. **AD** is a rod moving around **A.** **BE** revolves around **B,** and **LC** revolves around **C.** **LG** is a connecting rod;

Fig. 57.—Grasshopper Motion.

HD is a continuous lever to which **EB** and **GL** are pivoted at **E** and **G.** **HK** is a rod, the upper end of which is attached to **HD,** and the lower end to the cross head **K,** moving between the guides as shown. Suppose it is required to find the force **F** necessary to move **W,** the direction of motion being as shown by the feathered arrow.

GL. Taking first the two-force pieces, the line of action of the force in **GL** must be found. At the point **L** in **LG**, the external force acts as shown. The angle between **GL** and **LC** is closing. The friction force, therefore, acts to keep it open, and the line of action of the force through **L** and **G** passes to the right side of the friction circle at **L**. At the upper end, **G,** the external force acts as shown. To determine whether the angle between **GL** and **GH** is opening or closing, the instantaneous centers of these pieces and the relative angular velocities may be used. The piece **HG** has for its instantaneous center the point b, the intersection of **AD** and **BE** produced. **G** is then moving around b and the instantaneous center of **LG** is at the intersection of **G**b and **LC** produced, or at b''. The linear velocity of **G** is the same whether considered as a part of **HG** or of **LG** and the angular velocity of **GL** is greater than that of **HG** because its instantaneous radius **G**b'' is less than **G**b the instantaneous radius of **GH** at **G**. As **GL** is moving faster than **GH,** the angle between **GL** and **GH** is opening; the friction, therefore, acts to close this angle, and the line of action moves to the right side of the friction circle at the end **G**. The broken line then shows the line of action of the force **Q** acting in **LG.**

HK. The external force on **HK** at **H** acts upwards. The angle between **GH** and **HK** is opening; therefore the friction tends to close it, and the line of action of the force thus goes on the right side of the friction circle at **H**. As the motion is a straight line motion, at the lower end, **K,** there is no tendency for the rod **HK** to turn, and the line of action of the force **R** then passes through the center at **K,** and takes the direction as shown by the broken line. The friction force **T** from the guide then acts in the direction shown.

HD. As the forces on **HD** are in equilibrium, and as the lines of action of **DA** and **BE** are approximately in the direction of the lines **DA** and **BE,** we can draw an approximate force polygon as shown by the small diagram, *dehg,* giving us approximately the direction in which the forces **BE** and **DA** act. The force at **H** on **HD** acts downward, and the force at **G** acts upward. As **HD** is a four-force piece, the resultant of **H** and **G** must act approximately in the direction from their intersection to the intersection of **AD** and **BE** at b. Draw h, g, and their resultant, the

broken line. We can then complete the polygon for the piece **HD** by drawing the closing lines *d* and *e* parallel to **AD** and **BE**. The piece **DA** is therefore in tension, and the piece **BE** in tension.

AD. The external force at **A** acts downwards. **DA** is moving to the left, the friction force acts to the right, and the line of action of **DA** passes to the right of the friction circle. At **D**, the external force acts upward. The angle between **HD** and **DA** is closing; therefore, the friction force acts as shown by the curved arrow, and the line of action of the force at **D** passes to the right of the friction circle; therefore, the line of action of the force **AD** is shown by the broken line.

BE. The force on **BE** is a tension force. The external force at **E** acts as shown. The angle between **BE** and **HG** is opening; therefore, the friction force acts as shown, and the line of action at **E** goes to the bottom of the friction circle at **E**. At **B**, the external force acts as shown. **BE** is moving downwards and therefore the friction force acts upwards, and the line of action of the force **BE** goes to the upper side of the friction circle at **B**, and the broken line shows the line of action of the force in **BE**.

LCW. The lines of action of the force **LG** and the force **W** intersect at *c*. Both these forces act downward; therefore, the force at **C** acts upward as shown. As the piece is turning in the direction of the arrow, the friction must act as shown, and the line of action of the force through **C** must go to the left of the friction circle and pass through the point *c*.

The line of action of each of the forces acting on the mechanism is now determined. Having **W** known a line is drawn parallel to **W**, and from the ends of it a line parallel to **P** and **Q**. Through the ends of *q*, draw lines parallel to line of action of **R** and of **S**, which goes now from the intersection of **R** and **Q** to *b'*, instead of to *b*, and this gives us the force **R** required, at the cross-head **K**.

To determine the value of **F**, we draw lines parallel to **T** and **F** from the ends of *r*. The diagram now gives the relation between the force at **F** and the force at **W**.

To determine the forces acting on **BE** and **DA**, resolve the force **S** into components parallel to **U** and **V**. This determines the value of the forces acting in all the pieces of the mechanism.

Friction on the Teeth of Gears. When two gear wheels act together, as in Fig. 58, and the one **A** is the driver, and **B**

Fig. 58.—Wheels in Gear.

the follower, the contact between the teeth begins inside the pitch circle of **A**, before reaching the line of centers, and the contact ends inside the pitch circle of the wheel **B**, after passing the line of centers, at some point, as a for the beginning of contact, and b for the end of contact.
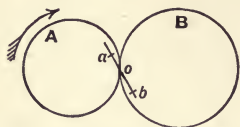
If the gears are cycloidal gears, instead of the path of the point of contact being a straight line ab, it is part of the describing circle for the flanks of the wheel **A** from the beginning at a to the line of centers at o, and part of the describing circle for the flanks of the wheel **B** from the line of centers at o to the end of contact at b. If the gears are involute ab is the path of the point of contact.

Approach. Fig. 59 represents the part of two teeth in contact at the beginning of contact. The path of the point of con-

Fig. 59.—Approach.

tact up to the line of centers, if the teeth are involutes, is indicated by the line drawn from a to d, and this line ad is the normal to the teeth at a whatever be the shape of the teeth. The point b on the driver reaches the point d at the same time that the point c on the follower reaches the point d. Evidently the length ac is longer than the length ab, and, as the teeth are in contact throughout the entire travel, it is evident that the face of the tooth of the driver must slide to the right over the face of the tooth of the follower, that is, the contact between the teeth is sliding, not rolling. Up to the line of centers, the tooth of the driver slides to the right over the tooth of the follower. If there is friction between the two teeth, ab and ac, the force exerted between the two teeth is not in the line
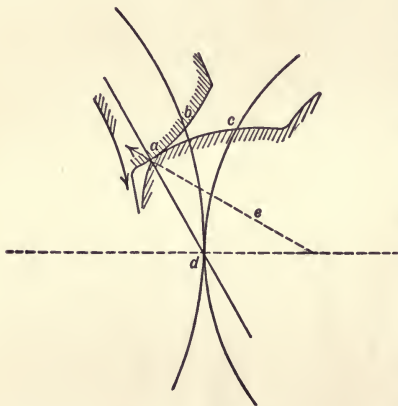
ad, which is normal to the teeth, but in a direction *ea*. The line *ae* makes an angle equal to the friction angle with the line *ad*, and the force on the driver acts from *e* to *a*, as shown by the arrow.

Recess. After passing the line of centers, the reverse is found to be the case. The termination of contact, that is, the point at which the point of the driver leaves the tooth of the follower is at *f*, Fig. 60. As the points *b* and *c* were in contact at
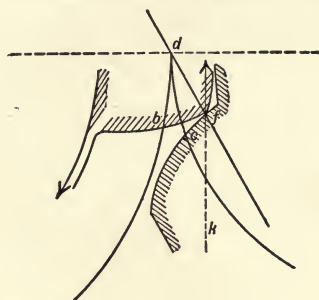
Fig. 60.—Recess.

d, and as evidently *fb* is longer than *fc*, while the contact passes from *d* to *f* the face of the tooth on the driver must have moved to the left over the face of the follower. If there is no friction between the teeth, then the line of the force acting between the teeth would have taken the direction *fd*, but, with friction, this line, instead of passing through *d*, takes the direction *kf*, the angle between *fk* and *df* being the friction angle, and the force exerted on the driver by the follower takes the direction as shown by the arrow.

Involutes. If, now, the teeth are involute teeth, and it is assumed that there are always two teeth in contact, and that the force is evenly divided between them, in Fig. 61, *af* represents the path of the point of contact of the two wheels, the left
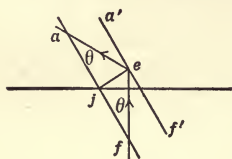
Fig. 61.—Effect of Friction.

being the driver. The two teeth are in contact at *a* and at *f*, the distance *af* being the pitch. The line of action between the teeth, because of friction, takes the direction *ea* at *a* and *fe* at *f*, and as each tooth has been assumed to carry an equal portion of the load or as the force is equally divided between the two teeth, the forces acting along the line *fe* and the line *ea* on the driver are equal, and act in the direction of the arrows shown in Fig. 61. The resultant of these two equal forces is evidently parallel to *af*, and the effect of friction between the teeth of gear wheels is to move the line of action of the teeth farther from the

center of the driver, while it remains parallel to its original direction. The distance *af* is the pitch, and the angle *efa* is the friction angle θ, and the distance *ej* equals

$$\frac{\text{Involute pitch}}{2} \times \tan \theta.$$

Geared Mechanism. Suppose three gear wheels are given centered at **A**, **B**, and **C**, as shown in Fig. 62. Let the arm **DA** carry a load **W** at the end of it; and the arm **CE** carry a load **P** at the end of it, both forces acting vertically downwards, and assume that there is no friction at **D** and at **E**. Assume, however, that
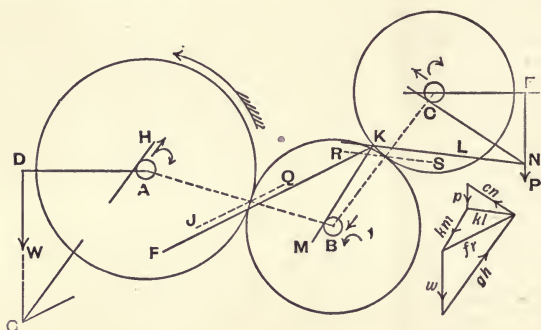
Fig. 62.—Geared Mechanism.

there is friction at the centers af the wheels **A**, **B** and **C**, and between the teeth of the wheels. Find the relation between the forces **W** and **P**, assuming the movement to be in the direction of the feathered arrow. The line of action between **A** and **B** takes the direction **JQ**, and, because of the friction, moves outward, parallel to itself, from the center of the driver **A** to the position **FK**. The wheel **B** turns in the opposite direction, and the line of action between **B** and **C** takes the direction as shown by the broken line **RS**, if there is no friction. If there is friction it moves out into the position **KL**. The wheel **A** with the arm **DA** is a three-force piece. As **W** is one of the forces and the force **KF** another, the third one must pass through their intersection. Both these forces act downwards; therefore the external force at **A** acts upward. The friction force acts as shown; and the third force which passes through the point **G** passes to the left of the friction circle, as shown at **GH**. The wheel **B** is a three-force piece, having a force acting along the line **FK**, a force acting along the line **KL**, and a third force acting at the support. Both

the forces **FK** and **KL** tend to raise **B,** and therefore the third force tends to force it downwards. **B** turns clockwise, therefore the friction force acts in the opposite direction, and the line of action of the force passes to the left side of the friction circle and through the intersection of **KL** and **FK,** and **KM** is therefore the direction of the third force acting on **B.** The wheel **C,** with its arm **CE,** is a three-force piece, having a force **P** and a force **KL** acting on it. These forces intersect at **N.** They tend to push the wheel **C** to the right and downward, and, therefore, the force acting through **C** and **N** tends to force the wheel upwards and to the left. The wheel is turning to the left, and therefore the friction force acts to the right, as shown, and the line of action of the force through **NC** passes to the lower side of the friction circle. The lines of action of all the forces acting on all the pieces have been determined, and, starting with **W** known, the force diagram can be drawn as shown in the figure, giving the value of the forces acting on all the various pieces of the mechanism. As the external forces acting on a piece of mechanism are in equilibrium, they must form a closed polygon, and knowing the direction of one of them, by following around the polygon, the direction of any other force can be determined. As all internal forces act in one direction on the driving piece and in the opposite direction on the driven piece, care must be taken not to read them into the external force polygon.

Thus, referring to Fig. 57, the force polygon for external forces is made up of $w\ f\ t\ u\ v\ p$, and knowing the direction of w the direction of the others can be read from the force polygon. In Fig. 62 the external forces are, p, km, w, gh, cn acting in the directions shown by the arrows. It is to be remembered that if there is no friction, the lines of action of these forces pass through the centers of the shafts, and through the intersection of the pitch circles on the line of centers.

Chains and Sprocket Wheels. A sprocket wheel **A** with the chain running over it is shown in Fig. 63. Assume that the direction of motion and of the forces acting on the two sides of the chain are as shown by the arrows. As long as there is no tendency for the links of the chain to move on themselves, the lines of action of the forces continue throughout the center of the length. As soon, however, as one link, **B,** for instance, is bedded

on the sprocket, the link **C** begins to turn in **B.** The external
force on the link **C**, at the top, is in the direction shown by the
small arrow.   The tendency of the link **C** is to move anti-clock-
wise, and the friction force is as shown by the curved arrow.   The
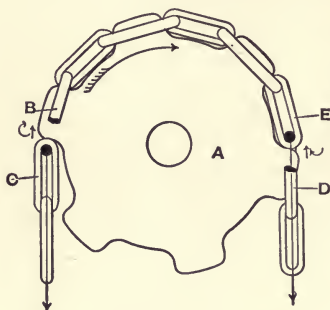
Fig. 63.—Sprocket Wheel.

line of action of the force between
B and **C** moves to the left, far
enough to be tangent to the fric-
tion circle, and the radius of the
friction circle is half the thickness
of the material of which the chain
is made times the sine of the fric-
tion angle.   On the opposite side,
as soon as the link **D** clears the
wheel **A**, it begins to turn on the
link **E**, and turns so that the line
of **D** and **E** tends to become
a straight line.   The external force
at the top of the link **D** acts in the direction of the small arrow
shown.   The friction force acts as shown by the curved arrow.
Therefore, the line of action of the force between **D** and **E** moves
to the left side of the friction circle.   If, then, a continuous
chain runs over a series of pulleys, the line of action on the
side on which the chain approaches the pulley moves away
from the line of centers, and on the side leaving the pulley, the

Fig. 64.—Several
Pulleys.

line of action moves toward the line of cen-
ters, as the chain leaving the pulley is doing
the work.
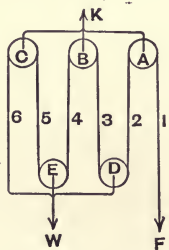
    Several Pulleys.   If a number of pulleys
are used, with the chain running continuously
over them all, the condition would be dia-
grammatically as shown in Fig. 64.   In this
figure it is assumed that there are three blocks
A, B and **C** on the same shaft, and two blocks
D and **E**, attached to a second shaft, to
which also the end of the chain is attached.

Evidently the . amount that can be raised on the hook **W** is
the sum of the pulls on the chains **2, 3, 4, 5** and **6**, while the
pull on the hook at **K** is the sum of all the pulls on **I, 2, 3,
4, 5** and **6.**   Friction causes the arm on which the force **F**

acts at the pulley **A** to decrease, and the arm on which the force **2** acts at the pulley **A** to increase. At **D**, the arm of **2** decreases, while that of **3** increases, so that generally on the top wheel, the pulls **l**, **3** and **5** act on shorter arms than the radius of the wheel, while **2**, **4** and **6** act on larger arms. At the lower wheels **D** and **E**, **2** and **4** act on shorter arms than the radius of the wheel, while **3** and **5** act on longer arms. Now, the arms are increased and decreased by the same amount, and this amount is the radius of the chain times the sine of the friction angle. It is therefore possible in a single diagram to determine the pull in each of the chains, and, by adding them together, to determine the amount of load that can be lifted by a given force.

In Fig. 65, suppose the distance from *a* to *b* to be the distance between the centers of the chain on the two sides of any

Fig. 65.—Effective Radii.

pulley. At *a* and *b* draw the friction circles for the chain, and around the center *c* draw the friction circle for the pin on which the wheel turns. Fiction reduces the driving arm to the distance marked **A**, and increases the arm on the driven side to the distance marked **B** in this figure.
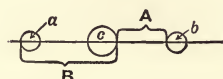
The sum of these two distances is the same as the original distance from *a* to *b*. At the wheel **A** of Fig. 64, the force **l** acts on the arm of **A** of Fig. 65, and the force **2** acts on the arm **B**, that is, **2** is less than **l**. At the wheel **D** of Fig. 64, the force **2** acts on the arm **A** of Fig. 65, while the force **3** acts on the arm **B**; therefore **3** is less than **2**.

Fig. 66.—Pulley Forces.

Similarly, **4** is less than **3**; **5** than **4**; and **6** than **5**.
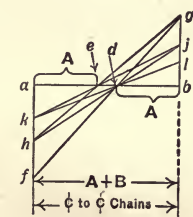
To get the value of these forces, draw in Fig. 66 a line *ab* whose length is the sum of **A** and **B**, Fig. 65, and mark on this line two points, each distant from one end the length **A** of Fig. 65. At one end, *a*, lay off the force *f*, and draw a vertical line through the other end of *ab*. Join *f* and *d*, and carry it on to *g;* then *bg* is the force **2**. Join *g* and *e*, and carry it on to *h;* then *ah* is the force **3**. Join *h* and *d*, and carry it on to *j;* then *jb* is the

force **4.** Join j and e, and carry it on to k; then ak is the force **5.** Join k and d, and carry it on to l; then bl is the force **6.** The force **W** is the sum of bg, ah, bj, ak, and bl, and the force **K** is greater than this by the force af. Without friction of course, the pull on **I, 2, 3, 4,** etc., is the same; therefore, the force **W** is ₅**F** and the force **K** is 6**F**.

Belt Friction. Two pulleys **A** and **B**, connected by a belt are represented in Fig. 67. The shaft **A** receives the power, which is

Fig. 67.—Belts.

transmitted through the belt to the wheel **B**, and thus to its shaft. If the direction of rotation is that shown, then the tension in the part **C** of the belt is greater than that in the part **D**. The amount of work transmitted is the difference in tension on the two sides times the distance traveled by the belt. It is evident that, as the amount of power transmitted increases, the difference between the tension on the two sides must become greater, until, finally, there is no tension on the part **D**, and the belt will slip, either on the wheel **A**, or on the wheel **B**.
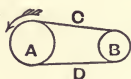
To determine the amount of force that can be transmitted by a belt without slipping, Fig. 68 represents a driving pulley with the belt carrying the tension **T″** at the one end, and **T′** at the other end, the pulley turning in the direction of the arrow. It is
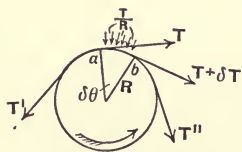
Fig. 68.—Pulley Friction.

assumed for convenience that the belt is one inch wide. The tension **T″** will then be greater than **T′**. At any point on the pulley as a, call the tension **T**, and, at a short distance from it, at b, suppose the tension to be **T** + d**T**. The difference in tension between the two points can only be due to the friction of the belt on the wheel. A circular band with a tension **T** in it is in exactly the same condition as is the shell of a boiler, and it must have a normal pressure acting on it, as in the case of a steam boiler. If the radius of the shell or pulley is **R**, then the internal pressure per square inch times the radius is equal to the tension per inch of breadth of the ring, or the normal pressure between the belt and the pulley at any point is $\dfrac{\mathbf{T}}{\mathbf{R}}$.

Call the angular distance between a and b, $d\theta$ radians. The pressure exerted by the belt on this portion of the pulley is $\mathsf{R}d\theta\dfrac{\mathsf{T}}{\mathsf{R}}$ and calling f the coefficient of friction, the friction exerted by the slipping belt is $\mathsf{R}d\theta\dfrac{\mathsf{T}}{\mathsf{R}}f$, and this must be the maximum value of $d\mathsf{T}$ or $d\mathsf{T} = \mathsf{R}d\theta\dfrac{\mathsf{T}}{\mathsf{R}}f$, or $\dfrac{d\mathsf{T}}{\mathsf{T}} = fd\theta$; or $\log\dfrac{\mathsf{T}''}{\mathsf{T}'} = f(\theta'' - \theta')$. Calling θ the angle between the point of contact where the belt comes on the pulley, and the point of contact where it leaves the pulley we have $\dfrac{\mathsf{T}''}{\mathsf{T}'} = \epsilon^{f\theta}$, which gives us the maximum ratio between the tension on the two sides of the belt. The difference of these tensions is the possible change in tension due to friction. If, to transmit the power, the difference in tension must be greater than this on the two sides, the belt must slip; if less, the belt will probably creep when carrying any large amount of power.

Bending and Twisting Moments. By the moment of a force is meant the product of the force by the distance from the point about which the moment is taken, and the distance is measured at right angles to the line of action of the force. The moment is called a bending moment or a twisting moment, depending on the effect that it has on the piece under consideration. If the arm of the moment, that is, the line at right angles to the direction of the force, lies in the axis of the piece, the moment is called a bending moment. If the arm of the moment is at right angles to the axis of the piece, and if the force is in a plane at right angles to the axis, the moment is called a twisting moment. If the arm is neither at right angles to nor in the line of the axis of the piece, the moment is neither a simple bending nor a simple twisting moment, but a combination of the two.

Moment of a Force. The moment of a force increases proportionately with the length of the arm. It is evident, therefore, that if two lines meet on the line of action of the force, or diverge from this point, the distance between these two lines parallel to the force will be, to some scale, the moment of the force.

Thus, in Fig. 69, if the three forces act as shown, the moment of the left-hand force may be represented by the distance between
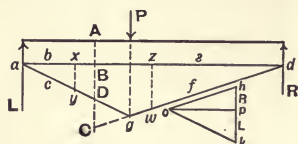
Fig. 69.—Moment Diagram.

the two lines ab and ac, measured parallel to the line of action of the force, that is, the moment of the force **L** is proportional to the distance xy. In the same way, the moment of the force **R** is proportional to the distance between

two other lines, de and df, and the moment of the force **R** is proportional to the distance zw, measured to some scale. As the moment of the force **R** taken under the load **P** is the same as the moment of the force **L** taken under the load **P**, if the three forces are in equilibrium, adg forms a closed triangle, but this closed triangle forms the equilibrium polygon. Therefore the equilibrium polygon represents the moments of the forces acting at the corners of the polygon.

Laying off the forces in the force polygon, and assuming that ad is perpendicular to the line of action of the forces, the distance op is the pole distance. The force hp is the force **R**, and force pk is the force **L**, and the force hk is the force **P**, in amount. The triangles zdw and oph are similar, and from the figure $\dfrac{zd}{zw} = \dfrac{op}{hp}$, as the angles at d and o are equal; or $op \times zw = zd \times hp$. As hp is **R**, the right-hand side of this equation is the moment of the force **R** on an arm zd, and, therefore the moment of the force is the distance zw times the pole distance op. That is, the moment of any combination of forces can be represented by an equilibrium polygon, and the actual moment is the height of the equilibrium polygon times the pole distance. Therefore, if one wants the moment at any point, after having an equilibrium polygon drawn, measure the distance zw in inches, multiply it by the distance op in inches, and multiply this product by the product of the scales of distance and force which were used in drawing the diagram.

Adding and Subtracting Moments. To add or subtract bending moments, they must be drawn to the same pole distance. At any point, **A** Fig. 69, the distance **BC** is the moment of the force **R**, with the pole distance op, and the distance **CD** is the

moment of the force **P**, with the pole distance *op*, and, therefore, at **A**, the resultant moment of the forces on the right of **A** is **CB** — **CD**, the difference being multiplied by the pole distance to get the moment in foot pounds or inch pounds.

Forces in One Plane. To draw the bending moment due to two forces, **P** and **Q**, Fig. 70, having the reactions unknown and acting in the line of **L** and
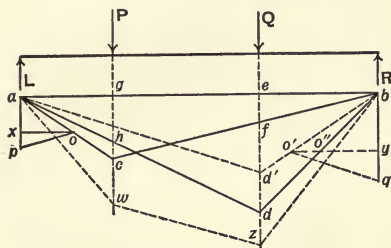
Fig. 70.—Adding Moments.

R, first draw the equilibrium polygon due to **P** alone, as shown by the triangle *abc*. This triangle may be any triangle, but it is more convenient to make *ab* perpendicular to the line of action of the forces. Lay off from *a* the force **P** to *p*, and draw a line through *p* to *o* parallel to *cb*. *o*, the intersection of *ao* and *po*, is the pole, and *ox* perpendicular to *ap* is the pole distance. To draw the bending moment due to **Q** to the same pole distance the reactions due to **Q** must be determined. First, draw *any* triangle *abd'*, as shown by the broken lines. Lay off from *b* to *q* the force **Q** to the same scale that was used for *ap*, and through *q* draw *qo'* parallel to *d'a*. Through *o'* draw *o'y* parallel to *ab*. Then *by* is the right reaction due to **Q**, and *yq* is the left reaction due to **Q**. The pole distance *o'y* is larger than the pole distance *ox*, and, therefore, the moments due to these two forces cannot be added directly from these triangles.

Laying off from *y* a distance *yo''* equal to *xo*, *o''* becomes the pole. Through *o''b* draw the line *bd* and complete the polygon by drawing the line *ad*. Then *abd* is an equilibrium polygon due to the force **Q** to the same pole distance that was used to draw the equilibrium polygon for **P**. At any point, as *e* the bending moment due to both **P** and **Q**, is the sum of the moments due to **P** and **Q** independently, as they are both in the same plane. Therefore, if the distances *ef* and *ed* be added, the resultant bending moment *ez* due to both forces is obtained. In the same way, at the point *g*, add together *gh* and *gc*, and lay off their sum from *g* to *w*. This will give the resultant bending moment at *g* due to both forces. Joining the points *a*, *w*, *z* and *b* gives *abzw* for

the bending moment due to the forces **P** and **Q**. This figure will be recognized as the same figure that was drawn before for the equilibrium polygon for the four forces, **P**, **Q**, **R** and **L**.

Two Forces Given. Having forces **P** and **Q**, Fig. 71, in the same plane draw the force polygon for **P** and **Q** and select any point *o* as a pole, draw a line *wz* between the forces **P** and **Q** to represent one side of the equilibrium polygon, and the lines *zb* and *wa* to represent the two additional sides of the equilibrium polygon, the vertical distance between *wz* and the line *aw* being the moment of the force **P**. If, instead of having two reactions, as in the preceding figure, **P** and **Q** are to be carried on a single
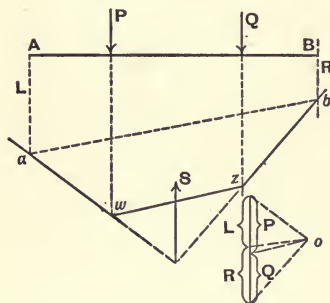
Fig. 71.—Two Forces Given.

reaction, this reaction must be placed at the point where *bz* and *aw* intersect, or at **S**. If, instead of carrying **P** and **Q** on a single reaction, one reaction is put at **A**, and one at **B**, the closing line of the equilibrium polygon is the broken line *ab*. The reactions are obtained by drawing through *o* a ray parallel to *ab*, and the distance between the upper ray and the one last drawn is the left reaction. Between the one last drawn and the lower ray is the right reaction, the reaction being determined by taking, in the force polygon, that portion of **P** plus **Q** lying between the inclined lines from *o*, the parallels to which meet at the point required.
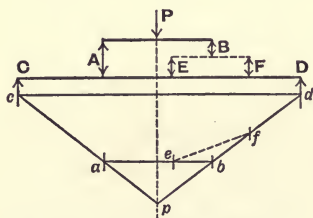
Secondary Beams. If the load, instead of being carried directly on the beam, is carried on a secondary beam, the bending moment on both beams can be determined as shown in Fig. 72. Suppose **P** to be the force acting on a secondary beam, **AB**, which in turn is supported on the main beam, **CD**. First assume that the load **P** acts directly on the main beam, and draw the equilibrium polygon, the triangle *cdp*. Through the points **A** and **B** draw vertical lines until they cut *cp* and *dp* respectively in *a* and *b*, and join *ab*.

Fig. 72.—Secondary Beams.

Then, the main beam has four loads acting on it, **C, A, B** and **D**, and the bending moment diagram is *cabd*. The bending moment diagram for the secondary beam is the small triangle *abp*, that is, the use of the secondary beam takes the corner *p* out of the equilibrium polygon from the main beam, and stops the sides of the equilibrium polygon, which previously extended to *p*, at the points *a* and *b* in the line of action of the loads that are put in by the use of the secondary beam.

If the right-hand support of the secondary beam is carried as shown by the dotted lines on **EF**, then the point *b* will disappear as far as the main beam is concerned, and, in its stead, two points, *e* and *f*, will be on the polygon, and the bending moment, as far as the main beam is concerned, will be *caefd*, and as far as the first secondary beam is concerned, will be *abp*, and as far as the secondary second beam is concerned, will be *ebf*.

Overhanging Secondary. If, instead of the secondary beam **EF** being as described above and shown in Fig. 72, it is an over-
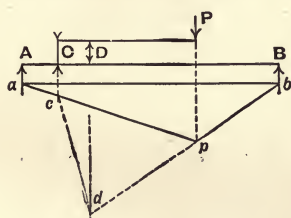
Fig. 73.—Overhanging Secondary.

hanging beam as shown in Fig. 73, the method of handling it is practically identical. First, draw the equilibrium polygon *abp*. Through **D** and **C** draw vertical lines, stopping the first one at *d* where it cuts *bp* produced, and stopping the second one at *c* in the line *ap*. Then join *cd*. The bending moment on the main beam is now *acdb*, and the bending moment on the secondary beam is *cdp*. The method of finding the forces acting at **C**

Fig. 74.—Overhanging Beams.

and **D** in Fig. 73, and at **A, B, E** and **F** in Fig. 72, is exactly the same as that shown in Fig. 71, and need not be repeated.

If the points of support, **C** and **D**, Fig. 73, are not within the distance **AB**, the method of procedure is exactly the same, although the appearance of the diagram changes. Draw the equilibrium polygon, Fig. 74, *abp*, as before. As the point *p* on the main beam is replaced by points on the lines **C** and **D**, continue

bp until it cuts either one of these vertical lines at *d*, and continue the other line *pa* until it cuts the other of these vertical lines in the point *c*, and join *cd*. Then, the bending moment on the main beam is *cabd*, and on the secondary beam is *cdp*.

Force Inclined to Beam. If the force, instead of being at right angles to the main beam, as shown in the last few figures, is at an angle to the main beam, as shown in Fig. 75, the bending
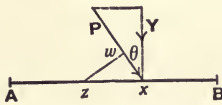
Fig. 75.—Inclined Force.

moment diagram differs from that already drawn. The force **P** has a component at right angles to the beam, and also a component lengthwise of the beam. The component lengthwise may be carried on a collar at **A**, in which case the left end of the beam **A***x* will be in tension; or by a collar at **B**, in which case the right end **B***x* will be in compression; or it may be distributed between collars at **A** and **B**. The bending moment at any point is produced by the vertical component of the force **P**. In the figure, the moment of the force **P** around any point, as *z*, is the force **P** times the distance *zw;* but the bending moment on the beam due to **P** is the force **Y** times the distance *zx*. From the two triangles, **P***x***Y** and *wzx*, we have $\dfrac{P}{Y} = \dfrac{zx}{zw}$, or $P \times zw = zx \times Y$.

The last term is the bending moment; therefore the moment of the force **P** about *z* is the same in amount as the bending moment on the beam **AB**, due to the force **P** at the point *x*. It is not, therefore, necessary to find the vertical component of **P**
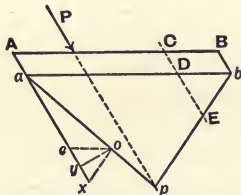
Fig. 76.—Bending Moment of Inclined Force.

to determine the bending moment, but the diagram can be drawn as shown in Fig. 76, and the moment measured directly as in any bending moment diagram.

Through **A** and **B**, Fig. 76, draw the reactions parallel to the force **P**, and draw any triangle having its vertices on the line of action of the three forces **A**, **B** and **P**. Lay off from *a* a distance *ax* equal to the force **P**, and draw a line from *x* parallel to the side *pb* of the equilibrium polygon. Then *o* is the pole, and the line *oy* at right angles to *ax* is the pole distance.

Draw a line *oc* through *o* parallel to *ab*. This line cuts *ax* in *c* determining the two reactions, *ac* being the left reaction and *cx* the right reaction.

The bending moment at any point, as, for instance, **C**, is the distance **DE** measured parallel to the force **P**, multiplied by the pole distance *oy*, and this product must be multiplied by the product of the scales to get the moment in foot pounds.

Inclined Force and Secondary Beam. If, instead of the inclined force being carried directly on the beam, it is carried on a secondary beam, as shown in Fig. 77, the bending moment dia-

Fig. 77.—Inclined Force, Secondary Beam.

gram is drawn as in Fig. 76, and the corner *p* is replaced by the corner *c* on *ap* produced, and *d* on *bp*, thus locating the closing line *cd*. The bending moment on the main beam is *acdb*, and on the secondary beam is *pcd*, the bending moment being measured in each case parallel to the force **P**, and the pole distance measured at right angles to it. It must be remembered that the piece **AB** or some part of it will have direct tension or compression in it.

Secondary Beam Perpendicular to Main Beam. If the secondary beam, in Fig. 77, instead of being inclined to the main beam **AB**, is at right angles to it as shown in Fig. 78, and if the force

Fig. 78.—Force Parallel to Beam.

P is parallel to the main beam, there must be, to hold it in equilibrium, a force equal and opposite to **P**, acting in the line of the beam. The two forces **P** then form a couple tending to turn the beam anti-clockwise, and, to hold it in equilibrium, a couple having the same moment, and tending to turn it clockwise, is required. Calling the forces **P** and **P'** and **Q** and **Q'**, although the two **P**'s are equal, as well as the two **Q**'s, the force polygon is drawn as shown below the beam. Taking any point , *o*, as a pole, knowing the value of **P** and **P'**, draw that side *vx* of the equilibrium polygon parallel to *oa* between the lines of action **P** and **P'**. Then, through each end of the line just drawn, draw lines parallel

to *ob*. It is worth remembering that whenever a couple acts there are always two parallel lines in the equilibrium polygon. The closing line of the equilibrium polygon will then pass through the points *s* and *t*. Drawing the ray *oc* through *o* parallel to *st* will give the value of the forces **Q** and **Q'**. The bending moment on the main beam is the vertical height of the bending moment diagram times the pole distance drawn from *o* perpendicular to **Q**, and the bending moment on the secondary beam is the horizontal height of the bending moment diagram times the pole distance from *o* perpendicular to **P**.

If, instead of assuming that the arm carrying **P** is an integral part of the main beam, it had been assumed that it was secured to the main beam by collars, there would be two additional bending forces on the main beam, acting one at either collar, and, in Fig. 79, is shown a construction of this type with the forces acting on it.
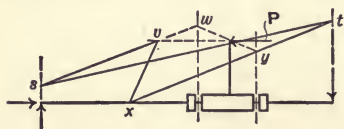
Fig. 79.—Secondary Beam and Parallel Force.

The bending moment on the secondary beam is the same as before. On the main beam, the bending moment is obtained by carrying the line *sv* until it cuts one of the new forces in *w*, and stopping the other line *tx* at *y*, and joining *w* and *y*, the closing line still remaining the same, *st*. The bending moment is now *tywst* for the main beam, and *svx* for the secondary beam. The pole distance for the bending moment on the main beam being the distance from *o* perpendicular to **Q** in Fig. 78, and from *o* perpendicular to **P** for the secondar*y* beam. As drawn in the last two figures, the bending moment on the secondary beam is the distance between those lines of the equilibrium polygon which meet on the line of action of **P**.

Forces not in one Plane. When two forces such as **P** and **Q** in Fig. 70 are not in the same plane, but are in planes inclined to each other, still acting, however, through the center of the beam, the reaction at the right end due to **Q** and that due to **P** are not in the same plane, but they can be replaced by a single force, and this single force will bend the beam from **R** to **Q**.

Similarly, the left reaction due to **P**, and that due to **Q** are not in the same longitudinal plane, but, as they act at the same

point and in the same transverse plane, they can be replaced by
a single force and this single force bends the beam from **L** to **P**,
but the bending is not in the same plane as the bending from **R**
to **Q**.
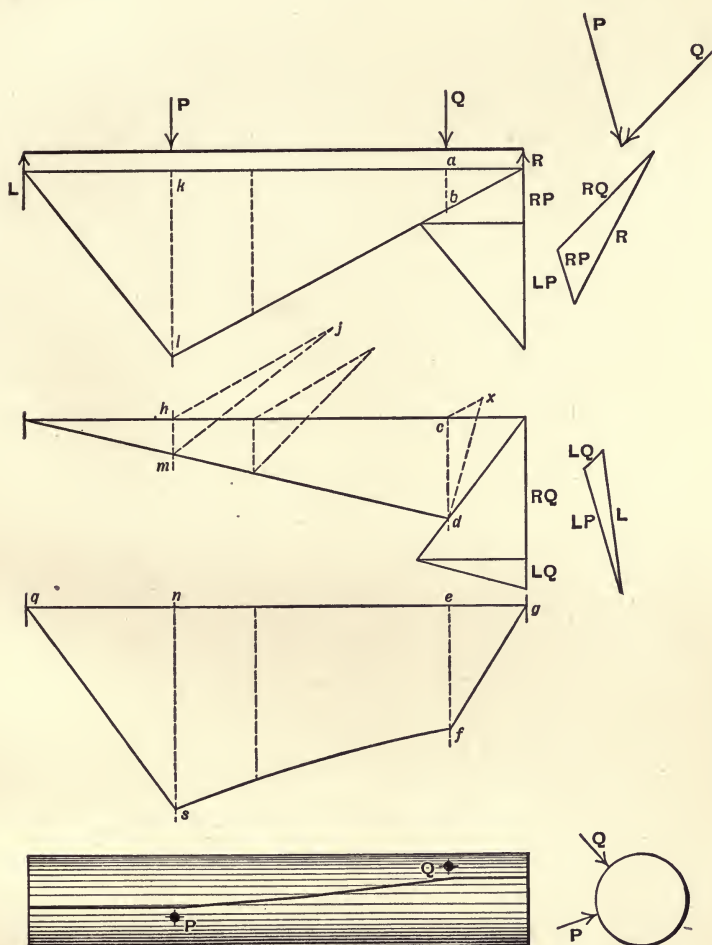
Fig. 80.—Bending Forces in Different Planes.

In Fig. 70, the method of finding the reactions due to both
P and **Q** has been shown, and that part of the construction is
repeated in Fig. 80, as though **P** and **Q** were in the plane of the
paper.

R is the resultant of **RP** and **RQ** acting in the opposite direction to the forces **P** and **Q**. The bending moment due to **R** is the resultant of that due to **RQ** and **RP**. Under the force **Q**, the distance *ab* is the bending moment due to **RP**, and *cd* is the bending moment due to **RQ**. From *c* lay off the line *cx* equal to *ba*, so that the angle *xcd* is the same as that between **RQ** and **RP**. Then *xd* is the bending moment under **Q** due to **R**, and that distance can be laid off from *e* to *f*. As the bending moment to the right of **Q** is that due to a single force **R**, the bending moment diagram has one side as represented by the line drawn from *f* to *g*.

Similarly, on the left-hand end, under **P**, lay off from *h* a distance *hj* equal to *kl*, the line *hj* being parallel to *cx*. Then *jm* is the bending moment under **P**. Laying off the distance *ns* equal to *jm* determines the bending moment at **P**, and as the bending moment to the left of **P** is caused by the single force **L**, which is the resultant of **LQ** and **LP**, a straight line drawn from *q* to *s* gives the bending moment from the left end of the beam up to the load **P**. Between **P** and **Q**, the bending moment is found point by point, in the same way, being the resultant of the two bending moments due to **P** and **Q**, and plotting the values, giving a curved line, such as is drawn in the lower figure, for the bending moment along the center of the beam. This curve is an hyperbola.

Compression in a Shaft. In dealing with a circular beam having the above loads on it, the point of maximum compression starting at the left end of the beam is on the top of the beam at a point opposite **L**, and these points lie in a straight line along the top of the beam from **L** to **P**. Starting from the right end, the point of maximum compression is on the top of the beam opposite **R**, but not in the same plane as at the left end. From **R** to **Q** the maximum compression points are in a straight line, but not in the same line as from **L** to **P**. Between **P** and **Q** the points of maximum compression are not in a straight line, but curve over the top of the beam between the straight line from **L** to **P** and that from **R** to **Q**. The bottom diagram of Fig. 80 shows this shaft looking down on it, the heavy line indicating the point of maximum compression at each cross section.

High Speed Shaft Forces. Forces of this class are forces such as exist in the crank shaft of an engine moving at high speed.

If m is that part of the weight of the connecting rod and crank arm which may be considered as acting at the crank pin, there is a centrifugal force acting through the center of the shaft $\dfrac{mr\omega^2}{g}$.

If two or more cranks not in the same plane are on the shaft, the conditions are identical with those above shown in Fig. 80.

Counter-balancing. As it is often desirable to know what weights must be put at particular points to counterbalance the effect of these centrifugal forces, the problem can be handled as shown in the last figure. If, in Fig. 80, **P** and **Q** are the two centrifugal forces to be balanced and **R** and **L** the lines of action in which the balance forces are to act, the amount of the balance forces must be **R** and **L** as determined in Fig. 80, and the direction must be parallel to **R** and **L** as shown in the small force triangles on the right.

General Method. When there are more than two forces, the problem can be solved more quickly in a little different way, the general method being worked out for the two forces and it may then be applied to any number.

In Fig. 81 are shown two cranks **A** and **B** at a given angle, with known centrifugal forces acting, and it is desired to know what forces, and where placed, will hold these forces in equilibrium, or whether any forces will do it. The forces are equal to $\dfrac{mr\omega^2}{g}$ in which m is the weight, ω is the angular velocity and r the radius at which m acts. The principles underlying the construction are: first, the force polygons in three planes at right angles to each other must close; and secondly, the equilibrium polygons in three planes at right angles must close if the moving body is to be in equilibrium in every direction.

Draw first a force diagram, looking at the end of the shaft. Starting at c, lay off the distance cd equal to the force **A**, and de equal to the force **B**, giving ce for the resultant force.

That this revolving shaft should be in equilibrium, it is necessary that the forces should be in equilibrium in three planes, at right angles to each other. Thus, looking endways at the cranks, the force polygon cde must close. Looking sideways at the cranks, in the direction ec, the equilibrium polygon must close, and in the direction at right angles to ec, the equilibrium polygon

must also close.   Resolving the forces along the line **OX** parallel
to *ec*, and **OY** perpendicular to *ec*, and revolving the **OX** compo-
nents about **O** into the line **OY'**, draw the equilibrium polygon
for the forces acting in the planes **OX** and **OY**, using the same pole
distance which we may assume to be any distance.   The equili-
brium polygon at right angles to *ec* has for sides the lines *hj*, *jk*
and *kl*.   As the lines *hj* and *kl* are parallel to each other there is
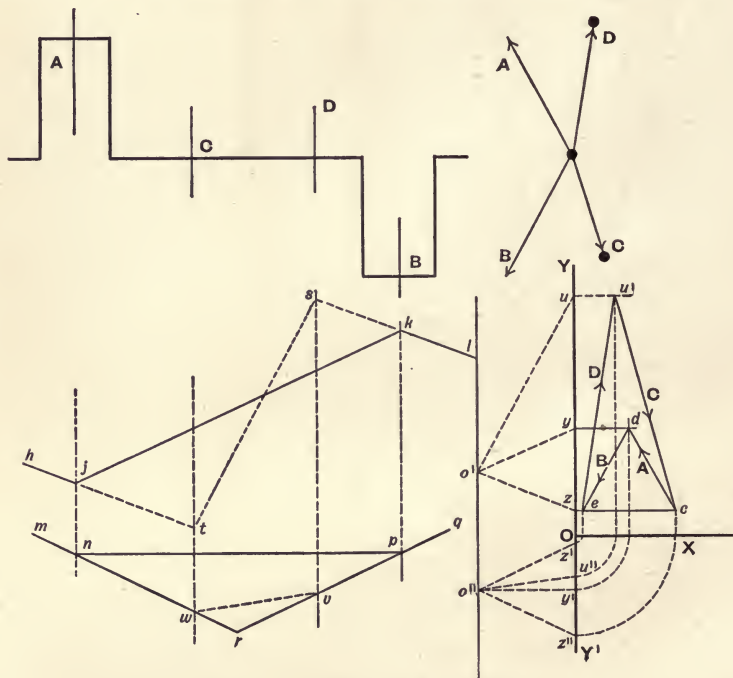
Fig. 81.—Balancing Two Cranks.

a couple acting on this shaft, practically in the plane of the paper,
which cannot be balanced by any single force.   In the plane
at right angles to this, the equilibrium polygon *mnpq* would close
at *r*, and a single force in this plane which is about at right
angles to the plane of the paper, would balance the horizontal
components of the forces **A** and **B**.

    If it is desired that these forces should be balanced in both
directions, suppose it is decided to put in weights in the lines **C**

and **D.** How much must they be, and what position must they occupy with respect to **A** and **B?** The upper equilibrium polygon closes on the line *st*, and drawing the ray through *o'*, parallel to *st*, gives us the line *o'u* for the ray going to the intersection of the two forces acting in the transverse planes of **C** and **D.** In the lower diagram, as the point *r* is to be replaced by two other points in *v* and *w*, *wv* is the closing line of the equilibrium polygon, and, drawing a line through *o''* to *u''*, we have the distance **O***u''* to determine the horizontal coordinate of the intersection of the forces that are to replace *ec;* therefore, the point *u'* is the point in the force polygon to which the lines from *e* and *c* must be drawn, or the line *eu'* is one of the balancing forces, and *u'c* is the other. The balancing forces are then those represented at **C** and **D** on the end view of the cranks. The values of these forces are *eu'* and *cu'* which together replace the resultant *ec*.

Our construction has been such that looking endwise at the shaft the force polygon *cdeu'c* closes and as these forces all pass through one point the equilibrium polygon would close. In a plane parallel to the paper and parallel to the shaft (**OY**) the force polygon *zyzuz* closes and the equilibrium polygon *jtsk* closes. In a plane parallel to the axis of the shaft and at right angles to the plane of the paper (**OX**) the force polygon *z''y'z'u''z''.* closes and the equilibrium polygon *npvw* closes. All the conditions for complete balance or equilibrium are met by this construction.

It is not necessary to lay off the forces proportional to $\dfrac{mr\omega^2}{g}$ as the $\dfrac{\omega^2}{g}$ has the same value for all the forces, and the diagram can be drawn so that the forces are proportional to *mr*, and, therefore the line *eu'* is proportional to the *mr* to be placed in the plane **D**, and *cu'* is proportional to the *mr* to be placed in the plane **C**, to balance the cranks in every direction.

Balancing Three Cranks. If, instead of having two cranks, there are three or more, exactly the same method of attacking the problem may be used. Fig. 82 shows the method of handling it, if one had three cranks, having equal values of *mr*, the cranks being set at 120 degrees.

Suppose the counter balances are to be placed in the lines

6

A and **B.** Draw the force polygon **123** as shown, and take the components along the lines **OY** and at right angles to it. Draw the two equilibrium polygons. The one parallel to the plane **OY** gives a couple, the closing line of which is the line *ab*. The equilibrium polygon parallel to the plane **OX** gives another couple. the closing line here being *a'b'*. The balancing forces are therefore the force *cd* and the force *dc*, or, putting them on the upper
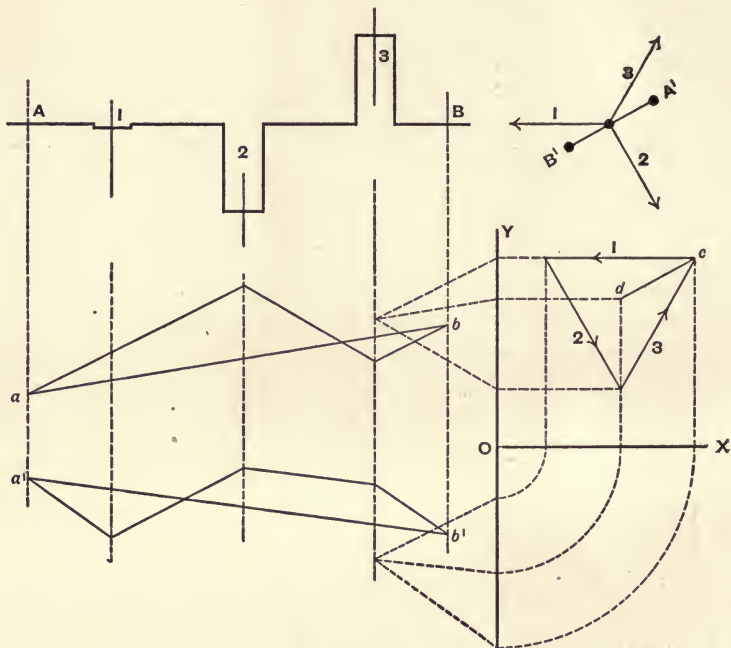
Fig. 82.—Balancing Three Cranks.

figure these two forces are equal to *cd* and occupy the position shown at **B'** and **A'**.

It must be remembered that, in the last three figures, the forces under consideration were all in the plane of the axis of the shaft, and produced bending only.

Overhung Crank. In the overhung crank shown in Fig. 83, having the bearings at **A** and **B**, and the point of application of the force at **C**, suppose, first, that the line of action of the force is in the plane of the crank. As far as the crank pin is concerned,

and as far as that portion of the shaft is concerned which lies to the left of the crank arm, they simply form a beam, acted on by a single overhanging force at **C**. In the equilibrium polygon, having forces acting in **A**, **B** and **C**, we have o for the pole, abc for the bending moment diagram, and op for the pole distance, the distance cp being the force **C**. The bending moment on the crank pin up to the point **E** is cye. Taking the crank arm, the bending moment at **E** on the crank
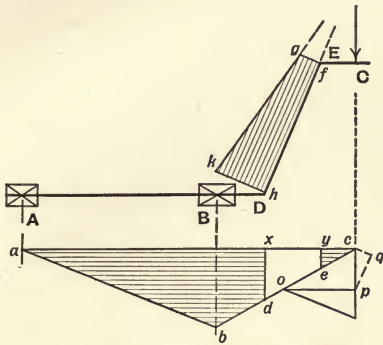
Fig. 83.—Overhung Crank. Force in Plane.

arm is the same as the bending moment at **E** on the crank pin, and therefore, a distance fg is laid off perpendicular to the arm and equal to ye. The bending moment at **D** on the crank arm is the same as the bending moment at **D** on the crank shaft, and therefore the distance hk, equal to xd, is laid off at **D**, and joining k and g gives us the diagram $fgkh$ for the bending moment on the crank arm. The lines kg and hf intersect on the line of action of **C**. In a case of this kind, the effect of putting in the crank arm has been to take the portion of the original bending moment diagram $xyed$ and stretch it out, so that the length is hf instead of xy, the side of the bending moment diagram ed becoming the straight line gk. The piece **DE** has direct compression in it, as the inclined component of the force cp acts in the direction of the axis of **DE**. The amount of the compression can be obtained by finding the component of cp parallel to **DE** as shown at pq.

Combination of Bending and Twisting Moments. In Fig. 84, a shaft is shown with bearings at **A** and **B**; on the shaft are mounted two gear wheels, **C** and **D**. An end view of the combination is shown on the right. Suppose the wheel **E**, in dotted lines, to be the driving wheel, turning the shaft in the direction shown by the arrow, and let **F** be the driven wheel. That part of the shaft between **B** and **C** has in it only the torsion due to the

friction at **B,** and the bend due to the force at **B.** The portion
of the shaft between **A** and **D** has in it the torsion due to the
friction at **A** and the bend due to the force at **A.** Between **C**
and **D** the torsion is due to the friction at **B** and the force exerted
by **E** on **C,** or, from the other end, the torsion between **D** and **C**
is the difference between that exerted by the friction at **A** and
that exerted by the wheel **F.**

Referring to the end view of the mechanism, the line of
action between **E** and **C,** taking friction into account, is the full
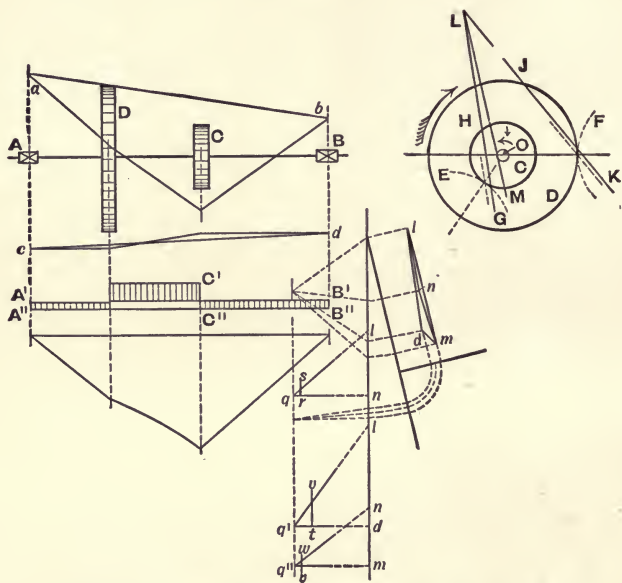
Fig. 84.—Combination of Bending and Twisting.

line **G H,** and the line of action between the wheel **D** and the wheel
F is the full line **J K.** The lines of action of these forces intersect
in **L.** The resultant of the forces acting at the bearings goes to
the left side of the friction circle, and passes through the point
L, thus the forces **KJ, G H** and **LM** act on the shaft, **LM** being the
resultant of the forces acting at **A** and **B.** Lay off the force
polygon for these forces as shown below the end view. Resolve
these forces parallel to, and at right angles to the force *lm,* and
draw the equilibrium polygon for these two sets of components

at right angles to each other. The line *ab* is the closing line of the equilibrium polygon in one direction, and *cd* is the closing line in the other direction, giving the point *n* from which to draw the forces which replace the force *lm* on the end view. Therefore, *ln* is the one reaction and *nm* is the other, *ln* being the right reaction and *nm* being the left one.

Bending and Torsion Diagram. The bending and torsion diagrams for the entire shaft can now be drawn. First the torsion diagram. Starting from the right-hand end, the force *ln* twists the shaft from **B** to **C** on an arm represented by the distance from the center of the shaft, **O**, to the line of action **LM**, or the radius of the friction circle.

Lay off the force *ln* on the vertical line that we have used for drawing our moment diagrams, and draw *nq* equal to the pole distance and at right angles to *ln*, and join *q* and *l*. From *q* lay off the radius of the friction circle to *r*. Then *rs* is the moment of *ln* on a radius *qr*. Drawing a rectangle whose height is *rs* gives the twisting moment **B′B″** on that portion of the shaft between **B** and **C**.

The force acting on **C**, tending to twist the shaft between **C** and **D**, acts in the opposite direction from the force *ln*. The arm on which the force on **C** acts is perpendicular from **O** to the line of action **HG**.

Lay off the force *ld* on the vertical line and mark the pole *q′* opposite *d* at right angles to *ld* and join *q′l*. From *q′* lay off a distance *q′t* equal to the distance from the center of the shaft to the line of action of **HG**, and *tv* is the torsion moment due to **C**. Laying this distance off from the base line, the torsion moment **C′C″** due to **C** is obtained. As the torsion moment between **C** and **D** is the difference found by subtracting that due to **B** from that due to **C**, continue the line through **B′**, and mark off as the torsion moment between **B** and **C** the difference of these two, as shown by the vertical hatching.

The torsion moment from **A** to **D** is obtained by laying off the force *nm*, marking *q″*, and laying off *q″v* equal to *qr*. Then *vw* is the torsion moment from **A** to **D**, and as it is constant throughout the whole length of this part of the shaft, the vertical hatched portion of the diagram represents this torsion.

As **C** is the driver, the work received by **C** is proportional

to **C'C″**; of this torsion a portion, **B'B″**, is used to overcome the friction at the end **B**, and a portion of it, **A'A″** is used to overcome the friction of the end **A**, and the remainder is given off by the wheel **D**. As the two equilibrium polygons are the bending moments in two planes at right angles to each other, the resultant bending moment can be obtained by combining the ordinates of these diagrams at right angles and the diagram shown at the bottom is the resultant bending moment on the shaft.

Force Perpendicular to Plane of Crank. If the crank is an overhung one as shown in Fig. 85, with the force acting at right

Fig. 85.—Overhung Crank. Force Perpendicular to Plane.

angles to the plane of the paper, the bending moment on the crank pin and shaft proper is given by the diagram *abc;* the portion *axdb* acts on the shaft **AD** and the portion *cye* acts on the part **CE** of the crank pin. As far as the part **ED** is concerned, the force at **C** both bends and twists it. Taking any point, *z*, in the piece **ED**, the moment of the force **P** is **P** times *z***C**. As *z***C** is neither in the axis of the piece **ED**, nor at right angles to it, this moment is neither pure bending nor pure torsion.

Suppose a perpendicular is dropped from **C** on the line **DE**, cutting it at *w*. No change is produced in a piece by putting in two equal and opposite forces at any point. Suppose such equal and opposite forces are placed at *w*, acting parallel to **P** and equal to **P**. Then, one of these forces, together with the force at **C**, produces a twisting couple on **ED**, and the amount of the twisting couple is the force **P** times the arm *w***C**. The remaining force at *w* tends to bend the piece **DE**. The bending moment is zero at *w*, and increases as we go towards **D**. As this is the moment of the force **P**, the divergence of the bending moment diagram from *w'd″* is the same as the divergence of the two lines *ac* and *cb* already drawn.

A diagram $w'd'd''$ can be drawn to represent the bending moment on **ED**. This diagram is such that the distance $d'd''$ is equal to the distance between ac and cb at a distance from c equal to $w'd''$. The bending moment diagram for the crank arm is therefore $e'd'd''e''$. This determines the bending moment in all the parts of the piece.

The twisting moment acting on **ED** is the force **P** acting on the arm $w\mathbf{C}$. Laying off the arm $w\mathbf{C}$ from c to o gives ot for the amount of the twisting moment on the arm **ED**, and as this is uniform throughout the whole length of this arm, the rectangle $o't't''o''$ is the torsion moment.

The part **AD** is twisted by the force **P** acting on the arm **CF**. Measuring from c a distance cf equal to **CF**, gives fg for the twisting moment on the part **AD** of the shaft. The twisting moment diagram is $b'g'g''b''$. The bending moment and the twisting moment on all parts of the piece are determined.

Crank Arm Perpendicular to Shaft. If the crank arm **ED** in Fig. 85, is at right angles to **AD**, as shown in Fig. 86, $axdb$ is the bending moment on the part **AD** of the shaft. xcd is the bending moment on the pin **EC**. The bending moment on **ED** is zero at **E**, and as it is the moment of the force **P**, the bending moment diagram $ed'd''$ for this portion of the shaft can be drawn.

Fig. 86.—Crank Arm Perpendicular to Shaft.

The twist on **ED** is the force **P** on the arm **EC**. Laying off the arm from e to c', we have $c'h$ as the twisting moment on **ED**, and, as it is uniform throughout the length of it, the twisting moment diagram can be drawn as shown on the left, the height of $h'c''$ being the same as hc'. The twisting moment of the part **AD** is the moment of the force **P** on the arm **ED**, and, in the figure, is the same as the distance $d'd''$, and this twisting moment diagram is as shown at the bottom of the figure.

It is worth noting that starting from c, the bending moment only increases from zero to xd. As soon as the corner is turned, this bending moment xd becomes the equal moment $h'c''$, twisting **ED**. Therefore, when the corner is turned at right angles the bending moment becomes a twisting moment. Now, the bending moment in **ED** increases from zero to an amount $d'd''$ at **D**, while the twisting moment at **D** is the same as it was at **E**. When the corner is again turned, the twisting moment $h'c''$ in the piece **ED** becomes the bending moment xd in the piece **AD** and the bending moment $d'd''$ in the piece **ED** becomes the twisting moment $x'd'''$ in the piece **AD**.

Center Crank. The crank shown in Fig. 87 is a center crank, with the torsion taken off from the left-hand end, the
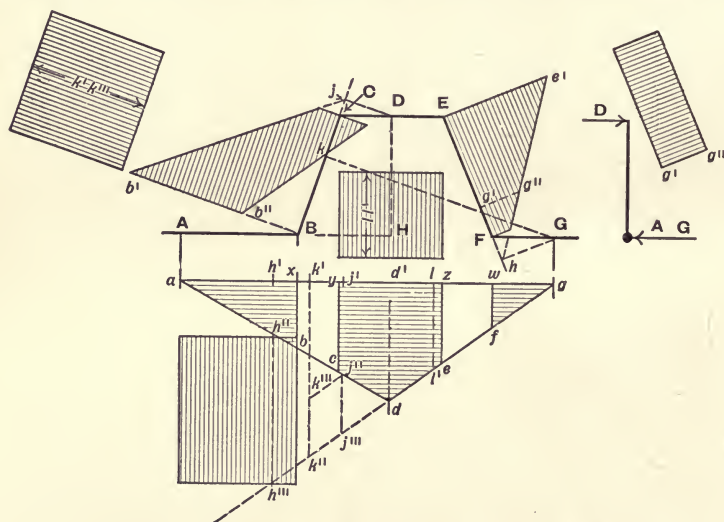
Fig. 87.—Center Crank. Force Perpendicular.

force on the crank being assumed as acting at right angles to the plane of the paper. The bending moment diagram for the part **AB** is axb; for the part **CDE** is $yzedc$, and for the part **FG** of the shaft the bending moment is wgf. The bending moment in the part **EF** is produced by the force **G**. It is zero where the perpendicular from **G** cuts **EF** at h, and as this bending moment is the moment of the force **G**, the divergence between the lines **E**h

and $e'h$ is the same as between ag and gd. The cross-hatched portion is then the bending moment on **EF**.

The bending moment on **CB** is the difference of the moments due to **D** and **G**. The moment due to **D** is zero at j, and the moment due to **G** is zero at k. Where the perpendicular from **D** cuts **BC** in j, draw a line jb', so that the divergence between jb', and j**B** is the same as the divergence between ad and gd extended, thus determining the moment of the force **D**. At any point between k and **B**, the forces **D** and **G** tend to bend the arm **CB** in opposite directions. Lay off from k a line kb'', such that the divergence of kb'' and **B**k is the same as the divergence between ag and gd, giving the moment of the force **G**. Below k, this must be subtracted from the moment of **D,** and above k it must be added to it. The cross-hatched portion of the diagram is the resultant bending moment on the arm **CD**.

As the torsion is taken off at the left end, there is no torsion, omitting friction, in the part **FG**. In the part **FE**, there is torsion due to the force **G** acting on the arm **G** h. Marking off a distance hg', equal to h**G**, the distance $g'g''$ represents the torsion on the piece **EF,** and the torsion diagram can then be drawn as shown at $g'g''$, and this is uniform throughout the entire length of **FE**.

The torsion in **CE** is produced by the force **G** acting on an arm **DH**. Laying off the distance **DH** from g to l, the distance ll' represents the torsion in **DE,** and the torsion diagram can be drawn as shown above.

In **CB**, the torsion is due to the force **G** which tends to turn it in one direction, and to the force **D** which tends to turn it in the opposite direction. Laying off from g a distance gk' gives $k'k''$ for the moment of the force **G**. Laying off from d' a distance $d'j'$ equal to **D**j gives the distance $j''j'''$ for the moment of the force **D**, and, subtracting this from $k'k''$ gives $k'k''$ ' for the twisting in the piece **CB**. The twisting moment diagram can be drawn as shown, and this is uniform throughout the entire length of the piece **BC**.

In the piece **AB**, as **G** acts through the line **AB**, it can have no twisting effect on it, and the only twist in **AB** is that due to the force **D** acting on the arm **DH**. Laying off from d' a distance $d'h'$ equal to **DH**, the distance $h''h'''$ is the twisting moment on

the part **AB** of the shaft.   Exactly the same method of proced-
ure can be followed with any combination of cranks, provided
the loads carried on the bearings can be definitely determined.

Load Inclined.   If the load, instead of being either in the
plane of the cranks, as in Fig. 83, or at right angles to it, as in
Fig. 87, is inclined, as in Fig. 88, the analysis is made as follows:—
Assume that the twist is taken off on the left.
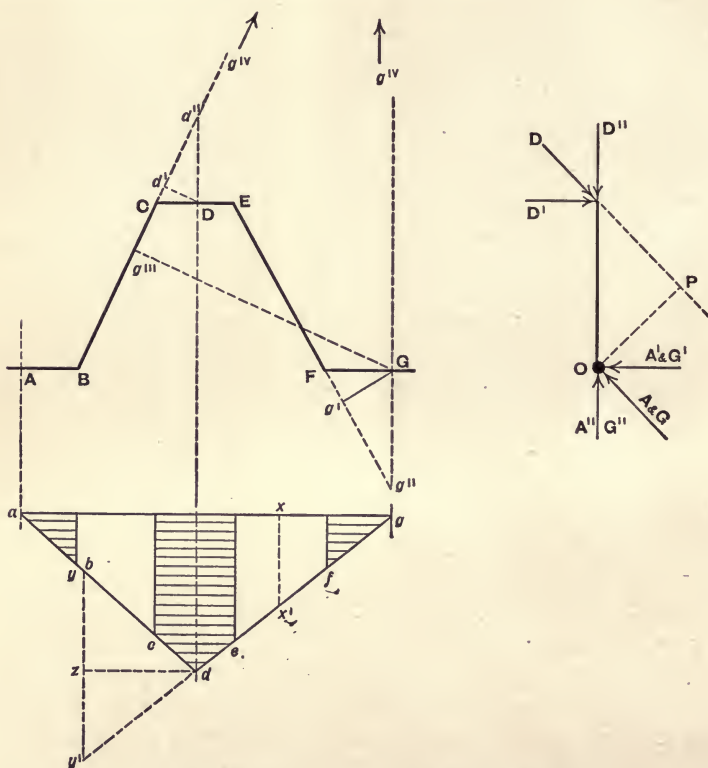
Fig. 88.—Center Crank.   Force Inclined.

Call **D** the force acting, **D′** its component at right angles to
the crank, and **D″** the component in the plane of the crank, the
reactions being **A′** and **G′** at right angles to the crank, and **A″** and
G″ in the plane of the crank.   The total bending moment in the
parts **AB**, **CE** and **FG** of the crank can be found by using the force

D and drawing the equilibrium polygon *adg*, and the parts used are the parts cross-hatched in the figure.

The total twist in the crank pin **CD** is that due to the total force **G** acting on the arm **OP**. As **G, A** and **D** were used in drawing the diagram, the twisting moment in the crank pin can be drawn of the height *xx'*, this being the moment of the force **G** on the arm **OP**.

The twist in the part **AB** is that due to the force **D** on the arm **OP**, and, from the above diagram, the value of this twist is as shown at *yy'*, the distance *dz* being **OP**.

Taking the piece **EF**, the force **G'** bends **EF**; the bending moment being zero at *g'* and the plane of the bend being at right angles to the plane of the paper. The force **G''** bends **EF**, its moment being zero at *g''*, and the plane of the bending moment being in the plane of the paper. The resultant of these two bending moments is the bending moment in **EF** due to **D**. The force **G''** compresses **EF**, and the amount of the compression is **G''** times the cosine of the angle G*g''g'*. The force **G'** twists **EF** and the arm of the twisting moment is *g'***G**; therefore **EF** has in it compression due to **G''**, twisting due to **G'**, and a resultant bending moment due to **G'** and **G''** acting at right angles to each other.

Referring to the piece **CB**, the force **G''** bends it in the plane of the paper, and this bending is zero at g^{IV}, the point in which **BC** and *g''***G** intersect. **D''** bends it in the opposite direction in the plane of the paper, and the zero of this bending moment is at *d''*. As the bending moments are in opposite directions the difference of these two is the bending moment in the plane of the paper. The force **G'** bends **CD** at right angles to the plane of the paper, the zero value of this bending moment occurring at *g'''*. The force **D'** bends **CB** at right angles to the plane of the paper, the amount of this bending moment being zero at *d'*.

In a plane at right angles to the plane of the paper there are two bending moments at any point as **B** acting in opposite directions; and the difference of these two bending moments is the resultant bending moment at right angles to the plane of the paper. The bending moment on the piece **CB** is therefore made up of two bending moments at right angles to each other, each of which is made up of two bending moments in their own planes.

The twist in **CB** is produced by **G'** and **D'** acting in opposite directions, **G'** acting on the arm **G**g''', and **D'** acting on the arm **D**d'. As these forces act in opposite directions, the twist on **CB** is the difference of the two.

In addition to this, the piece **CB** is in compression, due to the force **D''**, and in tension, due to the force **G''**, and the difference of these two will give the amount of compression in **CB**, or (**D''** — **G''**) times the cosine of the angle **C**d''**D** is the direct compression. This gives all the conditions existing in each of the members of this crank and shaft,

INDEX.

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